

Two-sided optimal bounds for Green functions of half-spaces for relativistic α -stable process

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Abstract

The purpose of this paper is to find optimal estimates for the Green function of a half-space of *the relativistic α -stable process* with parameter m on \mathbb{R}^d space. This process has an infinitesimal generator of the form $mI - (m^{2/\alpha}I - \Delta)^{\alpha/2}$, where $0 < \alpha < 2$, $m > 0$, and reduces to the isotropic α -stable process for $m = 0$. Its potential theory for open bounded sets has been well developed throughout the recent years however almost nothing was known about the behaviour of the process on unbounded sets. The present paper is intended to fill this gap and we provide two-sided sharp estimates for the Green function for a half-space. As a byproduct we obtain some improvements of the estimates known for bounded sets.

Our approach combines the recent results obtained in [5], where an explicit integral formula for the m -resolvent of a half-space was found, with estimates of the transition densities for the killed process on exiting a half-space. The main result states that the Green function is comparable with the Green function for the Brownian motion if the points are away from the boundary of a half-space and their distance is greater than one. On the other hand for the remaining points the Green function is somehow related the Green function for the isotropic α -stable process. For example, for $d \geq 3$, it is comparable with the Green function for the isotropic α -stable process, provided that the points are close enough.

Keywords: stable relativistic process, Green function, first exit time from a ball, tail function

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1 Introduction

In the paper we deal with some aspects of the potential theory of the *α -stable relativistic process*. That is a Lévy process on \mathbb{R}^d with a generator of the form

$$H_\alpha^m = mI - (m^{2/\alpha}I - \Delta)^{\alpha/2}, \quad 0 < \alpha < 2, \quad m > 0.$$

For $m = 0$ the operator above reduces to the generator of the *α -stable rotation invariant (isotropic) Lévy process* which potential theory was intensively studied in the literature.

For $\alpha = 1$ the operator

$$H_1^m = mI - (m^2I - \Delta)^{1/2}$$

plays a very important role in relativistic quantum mechanics since it corresponds to the kinetic energy of a relativistic particle with mass m . Generators of this kind were investigated for example by E. Lieb [19] in connection with the problem of stability of relativistic matter. An interested reader will find references on this subject e.g. in a recent paper [18].

Another reason that the operator H_α^m is an interesting object of study is its role in the theory of the so-called *interpolation spaces of Bessel potentials* and its application in harmonic analysis and partial differential equations (see, e.g. [23] and [15]). This theory is based on *Bessel potentials* defined as $J_\alpha = (I - \Delta)^{-\alpha/2}$. As Stein pointed out in his monograph [23], the Bessel potentials exhibit the same *local* behaviour (as $|x| \rightarrow 0$) as the Riesz potentials but the *global* one (as $|x| \rightarrow \infty$) of J_α is much more regular. In terms of the relativistic process the potential J_α is so-called 1-resolvent kernel of the semigroup generated by H_α^1 .

In the paper we consider the process killed on exiting the half-space $\mathbb{H} = \{x \in \mathbb{R}^d : x_d > 0\}$ and examine the behaviour of its Green function $G_{\mathbb{H}}(x, y)$. Contrary to the stable case a closed formula for that Green function is not known and seems to be a very challenging target. Recently in [5] an integral formula in terms of the Macdonald functions was found for $G_{\mathbb{H}}^m(x, y)$ - the m -resolvent kernel for \mathbb{H} . As proved in [5], for $d \geq 3$, the behaviour of the Green function is equivalent to the behaviour of the m -resolvent if $|x - y| \rightarrow 0$. Our main result establishes optimal bounds for the Green function of \mathbb{H} . To our best knowledge it is the first result of that type when optimal estimates for unbounded set (different than the whole \mathbb{R}^d) are derived.

At this point let us mention that the potential theory for bounded sets has been well developed during recent years (see [8], [22], [17], [14]). Under various assumptions of the regularity of a bounded open set D it was shown that the Green function of D was comparable with its stable counterpart. This comparison allowed to prove the relativistic potential theory shares most of the properties of the stable one if bounded sets are considered. Comparing the potential kernel for the stable process with the potential kernel for the relativistic process (see [20]) we can conclude that such a comparison of Green functions is not generally possible for unbounded sets. Since the relativistic potential kernel (for $d \geq 3$) is asymptotically equivalent (if $|x - y|$ is large) to that of the Brownian motion it may suggest that the Green function of \mathbb{H} , at least for some part of the range of x, y , is comparable with the Green function of \mathbb{H} for the Brownian motion. Our main result confirms that suggestion and we prove the comparability for points x, y being away from the boundary and with $|x - y| \geq 1$. For other points our bound is also optimal.

We also thoroughly examine the one-dimensional case and provide optimal estimates for the Green functions for bounded intervals taking into account their length. While for intervals of moderate length (say smaller than 1) we can use the well known results about comparability of stable and relativistic Green functions, for large intervals we rely on the estimates for half-lines obtained in this paper. Again we show that the Green functions for large intervals are comparable to the Brownian Green functions for most of the range.

The organization of the paper is as follows. In Section 2 we collect all definitions and preliminary results needed for the rest of the paper. The next section is basic for the paper. Here we prove the estimates for the Green function of $(0, \infty)$. Then in Section 4 we apply them to prove the optimal bounds for the tail function of the exit time from $(0, \infty)$ and some other properties of the exit times. These estimates will have a crucial role in examining multidimensional case which was accomplished in Section 5. We conclude the paper with exploring in the last section the one-dimensional case with regard to optimal estimates for bounded intervals.

2 Preliminaries

Throughout the paper by $c, C, C_1 \dots$ we denote nonnegative constants which may depend on other constant parameters only. The value of c or $C, C_1 \dots$ may change from line to line in a chain of estimates.

The notion $p(u) \approx q(u)$, $u \in A$ means that the ratio $p(u)/q(u)$, $u \in A$ is bounded from below and above by positive constants which may depend on other constant parameters only but does not depend on the set A .

We present in this section some basic material regarding the α -stable relativistic process. For more detailed information, see [22] and [7]. For questions regarding Markov and strong Markov property, semigroup properties, Schrödinger operators and basic potential theory, the reader is referred to [9] and [3].

We first introduce an appropriate class of subordinating processes. Let $\theta_\alpha(t, u)$, $u, t > 0$, denote the density function of the strictly $\alpha/2$ -stable positive standard subordinator, $0 < \alpha < 2$, with the Laplace transform $e^{-t\lambda^{\alpha/2}}$.

Now for $m > 0$ we define another subordinating process $T_\alpha(t, m)$ modifying the corresponding probability density function in the following way:

$$\theta_\alpha(t, u, m) = e^{mt} \theta_\alpha(t, u) e^{-m^{2/\alpha}u}, \quad u > 0.$$

We derive the Laplace transform of $T_\alpha(t, m)$ as follows:

$$E^0 e^{-\lambda T_\alpha(t, m)} = e^{mt} e^{-t(\lambda + m^{2/\alpha})^{\alpha/2}}, \quad \lambda \geq -m^{2/\alpha}. \quad (2.1)$$

Let B_t be the symmetric Brownian motion in \mathbb{R}^d with the characteristic function of the form

$$E^0 e^{i\xi \cdot B_t} = e^{-t|\xi|^2}. \quad (2.2)$$

Assume that the processes $T_\alpha(t, m)$ and B_t are stochastically independent. Then the process $X_t^{\alpha, m} = B_{T_\alpha(t, m)}$ is called the α -stable relativistic process (with parameter m). In the sequel we use the generic notation X_t^m instead of $X_t^{\alpha, m}$. If $m = 1$ we write $T_\alpha(t)$ instead of $T_\alpha(t, m)$ and X_t instead of X_t^1 . From (2.1) and (2.2) it is clear that the characteristic function of X_t^m is of the form

$$E^0 e^{i\xi \cdot X_t^m} = e^{mt} e^{-t(|\xi|^2 + m^{2/\alpha})^{\alpha/2}}.$$

Obviously in the case $m = 0$ the corresponding process is the standard (rotationally invariant or isotropic) α -stable process. X_t^m is a Lévy process (i.e. homogeneous, with independent increments). We always assume that sample paths of the process X_t^m are right-continuous and have left-hand limits ("cadlag"). Then X_t^m is Markov and has the strong Markov property under the so-called standard filtration.

From the form of the Fourier transform we have the following scaling property:

$$p_t^m(x) = m^{d/\alpha} p_{mt}^1(m^{1/\alpha}x). \quad (2.3)$$

In terms of one-dimensional distributions of the relativistic process (starting from the point 0) we obtain

$$X_t^m \sim m^{-1/\alpha} X_{mt},$$

where X_t denotes the relativistic α -stable process with parameter $m = 1$ and " \sim " denotes equality of distributions. Because of this scaling property, we usually restrict our attention to

the case when $m = 1$, if not specified otherwise. When $m = 1$ we omit the superscript "1", i.e. we write $p_t(x)$ instead of $p_t^1(x)$, etc.

Various potential-theoretic objects in the theory of the process X_t are expressed in terms of modified Bessel functions K_ν of the second kind, called also Macdonald functions. For convenience of the reader we collect here basic information about these functions.

K_ν , $\nu \in \mathbb{R}$, the modified Bessel function of the second kind with index ν , is given by the following formula:

$$K_\nu(r) = 2^{-1-\nu} r^\nu \int_0^\infty e^{-u} e^{-\frac{r^2}{4u}} u^{-1-\nu} du, \quad r > 0.$$

For properties of K_ν we refer the reader to [11]. In the sequel we will use the asymptotic behaviour of K_ν :

$$K_\nu(r) \cong \frac{\Gamma(\nu)}{2} \left(\frac{r}{2}\right)^{-\nu}, \quad r \rightarrow 0^+, \quad \nu > 0, \quad (2.4)$$

$$K_0(r) \cong -\log r, \quad r \rightarrow 0^+, \quad (2.5)$$

$$K_\nu(r) \cong \frac{\sqrt{\pi}}{\sqrt{2r}} e^{-r}, \quad r \rightarrow \infty, \quad (2.6)$$

where $g(r) \cong f(r)$ denotes that the ratio of g and f tends to 1. For $\nu < 0$ we have $K_\nu(r) = K_{-\nu}(r)$, which determines the asymptotic behaviour for negative indices.

The α -stable relativistic density (with parameter $m = 1$) can now be computed in the following way:

$$p_t(x) = \int_0^\infty e^t \theta_\alpha(t, u) e^{-u} g_u(x) du, \quad (2.7)$$

where $g_u(x) = \frac{1}{(4\pi u)^{d/2}} e^{-\frac{|x|^2}{4u}}$ is the Brownian semigroup, defined by (2.2).

We also recall the form of the density function $\nu(x)$ of the Lévy measure of the relativistic α -stable process (see [22]):

$$\nu(x) = \frac{\alpha}{2\Gamma(1 - \frac{\alpha}{2})} \int_0^\infty e^{-u} g_u(x) u^{-1-\alpha/2} du \quad (2.8)$$

$$= \frac{\alpha 2^{\frac{\alpha-d}{2}}}{\pi^{d/2} \Gamma(1 - \frac{\alpha}{2})} |x|^{-\frac{d+\alpha}{2}} K_{\frac{d+\alpha}{2}}(|x|). \quad (2.9)$$

In the case $0 < \alpha < 2$ we have the following useful estimates (see [22] for the proof of the first lemma):

Lemma 2.1. *There exists a constant $c = c(\alpha, d)$ such that*

$$\max_{x \in \mathbb{R}^d} p_t(x) \leq c(t^{-d/2} + t^{-d/\alpha}). \quad (2.10)$$

Lemma 2.2. *For any $t > 0$ and $x \in \mathbb{R}^d$ we have*

$$p_t(x) \leq c(d, \alpha) \left(g_t(x/\sqrt{2}) + t\nu(x/\sqrt{2}) \right)$$

and

$$p_t(x) \leq \frac{c(d)}{|x|^d}.$$

Proof. Notice that for $u, t > 0$,

$$\theta_\alpha(t, u) = t^{-2/\alpha} \theta_\alpha(1, t^{-2/\alpha} u) \quad \text{and} \quad \theta_\alpha(1, u) \leq cu^{-1-\alpha/2}.$$

Hence

$$\theta_\alpha(t, u) \leq ctu^{-1-\alpha/2}, \quad t, u > 0. \quad (2.11)$$

Using (2.7) we obtain for $t \geq 1$,

$$\begin{aligned} p_t(x) &\leq e^{-\frac{|x|^2}{8t}} (4\pi)^{-\frac{d}{2}} e^t \int_0^{2t} \theta_\alpha(t, u) e^{-u} u^{-d/2} du \\ &\quad + ct \int_{2t}^\infty g_u(x) e^{-\frac{u}{2}} u^{-1-\alpha/2} du \\ &\leq e^{-\frac{|x|^2}{8t}} (4\pi)^{-\frac{d}{2}} e^t \int_0^\infty \theta_\alpha(t, u) e^{-u} u^{-d/2} du \\ &\quad + ct \int_0^\infty g_u(x) e^{-\frac{u}{2}} u^{-1-\alpha/2} du \\ &= e^{-\frac{|x|^2}{8t}} p_t(0) + ct\nu(x/\sqrt{2}), \end{aligned}$$

where we used (2.8) in the last line. Moreover by Lemma 2.1 we can estimate

$$e^{-\frac{|x|^2}{8t}} p_t(0) \leq cg_t(x/\sqrt{2}), \quad t \geq 1.$$

This completes the proof of the first estimate for $t \geq 1$.

Next, for $t \leq 1$, applying (2.7), (2.11) and (2.8) we arrive at

$$p_t(x) \leq ct \int_0^\infty g_u(x) e^{-u} u^{-1-\alpha/2} du = ct\nu(x) \leq ct\nu(x/\sqrt{2}),$$

which complete the proof the first inequality.

The second bound is true for the transition density of any subordinated Brownian motion. Indeed let us observe that for any $t > 0$ and $x \in \mathbb{R}^d$,

$$g_t(x) \leq \left(\frac{d}{2\pi} \right)^{d/2} e^{-\frac{d}{2}|x|^2}.$$

Hence by subordination

$$p_t(x) = E g_{T_\alpha(t)}(x) \leq \left(\frac{d}{2\pi} \right)^{d/2} e^{-\frac{d}{2}|x|^2}.$$

□

The standard reference book on general potential theory is the monograph [3]. For convenience of the reader we collect here the basic information with emphasis on what is known (and needed further on) about the α -stable relativistic process.

In general potential theory a very important role is played by λ -resolvent (potential) kernels, $\lambda > 0$, which are defined as

$$U_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t(x - y) dt, \quad x, y \in \mathbb{R}^d.$$

If the defining integral above is finite for $\lambda = 0$, the corresponding kernel is called a potential kernel and will be denoted by $U(x, y)$. For the relativistic process the potential kernel is well defined for $d \geq 3$ but contrary to the stable or Brownian case it is not expressible as an elementary function. Recall that for the isotropic α -stable process the potential kernel is equal to $C|x - y|^{\alpha-d}$ for $d > \alpha$ and for the Brownian motion it is $C|x - y|^{2-d}$ for $d \geq 3$, where C 's are appropriate constants. One can prove that the relativistic potential kernel could be written as a series involving the Macdonald functions of different orders but this formula does not seem very useful. Nevertheless the asymptotic behaviour of the potential kernel was established in [13], [20].

$$U(x - y) \approx |x - y|^{\alpha-d}, |x - y| \leq 1; \quad U(x - y) \approx |x - y|^{2-d}, |x - y| \geq 1. \quad (2.12)$$

Note that they suggest that the process locally behaves like a stable one and globally like a Brownian motion. Despite the fact we do not know any simple form for the potential kernel, a formula for the 1-potential kernel is known (e.g. see [5]):

$$U_1(x) = C(\alpha, d) \frac{K_{(d-\alpha)/2}(|x|)}{|x|^{(d-\alpha)/2}},$$

where $C(\alpha, d) = \frac{2^{1-(d+\alpha)/2}}{\Gamma(\alpha/2)\pi^{d/2}}$.

The *first exit time* of an (open) set $D \subset \mathbb{R}^d$ by the process X_t is defined by the formula

$$\tau_D = \inf\{t > 0; X_t \notin D\}.$$

The basic object in potential theory of X_t is the λ -harmonic measure of the set D . It is defined by the formula:

$$P_D^\lambda(x, A) = E^x[\tau_D < \infty; e^{-\lambda\tau_D} \mathbf{1}_A(X_{\tau_D})].$$

The density kernel of the measure $P_D^\lambda(x, A)$ (if it exists) is called the λ -Poisson kernel of the set D . If $\lambda = 0$ the corresponding kernel will be denoted by $P_D(x, z)$ called *Poisson kernel* of the set D .

Another fundamental object of potential theory is the *killed process* X_t^D when exiting the set D . It is defined in terms of sample paths up to time τ_D . More precisely, we have the following "change of variables" formula:

$$E^x f(X_t^D) = E^x[t < \tau_D; f(X_t)], \quad t > 0.$$

The density function of transition probability of the process X_t^D is denoted by p_t^D . We have

$$p_t^D(x, y) = p_t(x - y) - E^x[t > \tau_D; p_{t-\tau_D}(X_{\tau_D} - y)], \quad x, y \in \mathbb{R}^d. \quad (2.13)$$

Obviously, we obtain

$$p_t^D(x, y) \leq p_t(x, y), \quad x, y \in \mathbb{R}^d.$$

p_t^D is a strongly contractive semigroup (under composition) and shares most of properties of the semigroup p_t . In particular, it is strongly Feller and symmetric: $p_t^D(x, y) = p_t^D(y, x)$.

The λ -potential of the process X_t^D is called the λ -Green function and is denoted by G_D^λ . Thus, we have

$$G_D^\lambda(x, y) = \int_0^\infty e^{-\lambda t} p_t^D(x, y) dt.$$

If $\lambda = 0$ the corresponding kernel will be called *Green function* of the set D and denoted $G_D(x, y)$.

Integrating (2.13) we obtain for $\lambda > 0$,

$$G_D^\lambda(x, y) = U_\lambda(x, y) - E^x e^{-\lambda\tau_D} U_\lambda(X_{\tau_D}, y).$$

Suppose that $D_1 \subset D_2$ are two open sets. By the Strong Markov Property

$$\begin{aligned} G_{D_2}^\lambda(x, y) - G_{D_1}^\lambda(x, y) &= E^x [e^{-\lambda\tau_{D_1}} U_\lambda(X_{\tau_{D_1}}, y) - e^{-\lambda\tau_{D_2}} U_\lambda(X_{\tau_{D_2}}, y)] \\ &= E^x [\tau_{D_1} < \tau_{D_2}; e^{-\lambda\tau_{D_1}} (U_\lambda(X_{\tau_{D_1}}, y) - e^{-\lambda\tau_{D_2} \circ \theta_{\tau_{D_1}}} U_\lambda(X_{\tau_{D_2}}, y))] \\ &= E^x [\tau_{D_1} < \tau_{D_2}; e^{-\lambda\tau_{D_1}} (U_\lambda(X_{\tau_{D_1}}, y) - E^{X_{\tau_{D_1}}} e^{-\lambda\tau_{D_2}} U_\lambda(X_{\tau_{D_2}}, y))] \\ &= E^x [\tau_{D_1} < \tau_{D_2}; e^{-\lambda\tau_{D_1}} G_{D_2}^\lambda(X_{\tau_{D_1}}, y)]. \end{aligned} \quad (2.14)$$

The main purpose of the present paper is to obtain sharp estimates for the Green function for $D = \mathbb{H} = \{x \in \mathbb{R}^d : x_d > 0\}$. The investigation of Green functions of the relativistic process for unbounded sets seems not to be treated in the literature. For bounded sets there many results obtained in recent years showing that the Green functions for open bounded sets under some assumptions about regularity of their boundary are comparable to their stable counterparts in \mathbb{R}^d , $d > \alpha$ ([22], [8], [17]). That is, for $x, y \in D$,

$$C(D)^{-1} G_D^{stable}(x, y) \leq G_D(x, y) \leq C(D) G_D^{stable}(x, y), \quad (2.15)$$

where G_D^{stable} is the corresponding Green function for the isotropic stable process and $C(D)$ is a constant usually dependent on $\text{diam}(D) = \sup_{x, y \in D} |x - y|$. Unfortunately in all known general bounds of the above type the dependence on the set D in the constant $C(D)$ is not very clear and $C(D)$ grows to ∞ with $\text{diam}(D)$. The constant also depends on some other characteristics of D as e.g. Lipschitz characteristic of D when D is a Lipschitz set. Therefore it is not possible to use well known exact formulas or estimates for the stable Green functions of regular sets as half-spaces, balls or cones to derive the corresponding optimal estimates for the relativistic process. Even for balls the constants grow to ∞ and (2.15) does not yield any estimate for a half-space in the limiting procedure.

Now suppose that D is a bounded set with a $C^{1,1}$ boundary. It is well known that there is a $\rho > 0$ such that for each point $z \in \partial D$ there are balls $B_z \subset D$, $B_z^* \subset D^c$ of radius ρ such that $z \in \overline{B_z} \cap \overline{B_z^*}$. Denote by $\rho_0 = \rho_0(D)$ the largest ρ having the above property. Finally let $\gamma = \text{diam } D / \rho_0$. However not explicitly stated, the following bound can be deduced from the results proved in [22], for $d > \alpha$, $x, y \in D$:

$$\begin{aligned} C_1(\gamma) C(\text{diam}(D))^{-1} G_D^{stable}(x, y) &\leq G_D(x, y) \\ &\leq C_2(\gamma) C(\text{diam}(D)) G_D^{stable}(x, y), \end{aligned} \quad (2.16)$$

where the constant C can be chosen in such a way that $C(\text{diam}(D)) = 1$ for $\text{diam}(D) \leq 1$ and $C(\text{diam}(D))$ increases with $\text{diam}(D)$. With some extra effort one can prove that the growth is polynomial. The constants $C_1(\gamma), C_2(\gamma)$ can be chosen as continuous with respect to γ . Note that if D is a ball than we can take absolute constants (depending only on α and d) instead of $C_1(\gamma), C_2(\gamma)$.

Hence for "smooth" sets with small or moderate diameter the estimate (2.16) is very satisfactory. For example for balls of small or moderate diameter we obtain very precise estimates using well known results for the isotropic stable process. However, in the case of balls of large size, it would be very interesting to find optimal estimates of the relativistic Green function. Our main result provides optimal estimates for the Green function of the half-space \mathbb{H} . Also we found optimal estimates for intervals in \mathbb{R} . Despite the fact we do not examine Green functions for balls in higher dimensional spaces we provide very precise estimates of the expected first exit time from a ball.

Now we define harmonic and regular harmonic functions. Let u be a Borel measurable function on \mathbb{R}^d . We say that u is *harmonic* function in an open set $D \subset \mathbb{R}^d$ if

$$u(x) = E^x u(X_{\tau_B}), \quad x \in B,$$

for every bounded open set B with the closure $\overline{B} \subset D$. We say that u is *regular harmonic* if

$$u(x) = E^x [\tau_D < \infty; u(X_{\tau_D})], \quad x \in D.$$

As a result of (2.16) we obtain the following version of the Boundary Harnack Principle (for details see [22] or [14] in the one-dimensional case).

Theorem 2.3. *[BHP] Let D be a bounded set with a $C^{1,1}$ boundary. Suppose that $\text{diam } D \leq 4$ and $\rho_0(D) \geq 1$. Let $z \in \partial D$. If f is a non-negative regular harmonic function on D and $f(x) = 0$, $x \in B(z, 1) \cap D^c$. Then*

$$f(x) \approx f(x_0) \delta_D(x)^{\alpha/2}, \quad x \in B(z, 1/2),$$

where $\delta_D(x) = \text{dist}(x, \partial D)$ and $x_0 \in D$ such that $\delta_D(x_0) = 1$.

For the purpose of this paper we state the following specialized form of BHP which can be easily deduced from Theorem 2.3.

Lemma 2.4. *Let $\mathbb{H} \ni \mathbf{1} = (0, \dots, 0, 1)$ and let $F = B(0, \sqrt{2}) \cap \mathbb{H}$. Suppose that f is a regular nonnegative harmonic on F such that $f(x) = 0$, $x \in \mathbb{H}^c$. Then for every $x \in B(0, 1) \cap \mathbb{H}$ we have*

$$f(x) \approx f(\mathbf{1}) x_d^{\alpha/2}.$$

Assume that $R \geq 2$. Let $D = B(0, R)$, $z_0 = (0, \dots, 0, R)$ and $x_0 = (0, \dots, 0, R-1)$. Let $F = B(z_0, 2) \cap D$. Suppose that f is regular nonnegative harmonic on F such that $f(x) = 0$, $x \in D^c$. Then for every $x \in B(z_0, 1) \cap D$ we have

$$f(x) \approx f(x_0) (R - |x|)^{\alpha/2}.$$

As mentioned above, the one-dimensional case for intervals was treated recently in [14] and since we will need it in the next section we present it in a convenient form of the estimate of the Poisson kernel. Actually in [14] it was shown that the Green function of $(0, R)$ is comparable with the Green function of the corresponding stable process (with uniform constant for $R \leq 3$). By standard arguments (see [22]) this implies the lemma below.

Lemma 2.5. *Assume that $d = 1$ and $0 < R \leq 3$. Let $D = (0, R)$. Then*

$$P_D(x, z) \approx \frac{(x(R-x))^{\alpha/2}}{(R(z-R))^{\alpha/2}(z-x)} e^{-z}, \quad x \in D, \quad z > R.$$

This implies that

$$P^x(X_{\tau_D} > R) \approx (x/R)^{\alpha/2}, \quad x \in D$$

and

$$E^x[X_{\tau_D} > R; X_{\tau_D}] \approx (x/R)^{\alpha/2}((R-x)^{\alpha/2} + x), \quad x \in D.$$

We also have that

$$E^x \tau_D \approx (x(R-x))^{\alpha/2}, \quad x \in D.$$

Obtaining any exact formulas for the Green function or the Poisson kernel even for regular sets seems to be a very hard task but in the recent paper [5] the formulas for the 1-Poisson and 1-Green function of \mathbb{H} were described explicitly in terms of the Macdonald functions:

Theorem 2.6. *Let*

$$E^x[e^{-\tau_{\mathbb{H}}}, X_{\tau_{\mathbb{H}}} \in du] = P_{\mathbb{H}}^1(x, u)$$

be the 1-Poisson kernel for \mathbb{H} . Then we have

$$P_{\mathbb{H}}^1(x, u) = 2 \frac{\sin(\alpha\pi/2)}{\pi} (2\pi)^{-d/2} \left(\frac{x_d}{-u_d} \right)^{\alpha/2} \frac{K_{d/2}(|x-u|)}{|x-u|^{d/2}},$$

where $u_d < 0 < x_d$. Let $G_{\mathbb{H}}^1(x, y)$ be the 1-Green function for \mathbb{H} then for $x, y \in \mathbb{H}$,

$$G_{\mathbb{H}}^1(x, y) = \frac{2^{1-\alpha}|x-y|^{\alpha-d/2}}{(2\pi)^{d/2}\Gamma(\alpha/2)^2} \int_0^{\frac{4x_dy_d}{|x-y|^2}} \frac{t^{\frac{\alpha}{2}-1}}{(t+1)^{d/4}} K_{d/2}(|x-y|(t+1)^{1/2}) dt.$$

Moreover,

$$\begin{aligned} \int_{\mathbb{H}} G_{\mathbb{H}}^1(x, y) dy &= 1 - E^x e^{-\tau_{\mathbb{H}}} \\ &= \frac{1}{\Gamma(\alpha/2)} \int_0^{x_d} t^{\alpha/2-1} e^{-t} dt, \quad x \in \mathbb{H}. \end{aligned} \tag{2.17}$$

This result will be very useful in our analysis since, as shown in [5] the behaviour of the Green function $G_{\mathbb{H}}(x, y)$ could be described in terms of the 1-Green function $G_{\mathbb{H}}^1(x, y)$ when x and y are close enough.

One of our main tools in establishing the upper bounds of the Green function will be estimates for the tail function $P^x(\tau_{\mathbb{H}} > t)$. We start with the following lemma taken from the Master Thesis of the first author [13].

Lemma 2.7. *There is a constant C such that*

$$P^x(\tau_{\mathbb{H}} > t) \leq C \frac{x_d + \ln(t+1)}{t^{1/2}}, \quad t \geq 1, \quad x_d > 0. \tag{2.18}$$

Proof. Let $Y_t = X_t^{(d)}$, where $X_t = (X_t^{(1)}, \dots, X_t^{(d)})$. By the symmetry of the random variable Y_t we obtain

$$\begin{aligned} P^x(\tau_{\mathbb{H}} > t) &= P^x(\inf_{s \leq t} Y_s > 0) \\ &= P^0(\inf_{s \leq t} (-Y_s + x_d) > 0) = P^0(\sup_{s \leq t} Y_s < x_d). \end{aligned}$$

Using a version of the Lévy inequality ([2], Ch.7, 37.9) we have for any $\varepsilon, y > 0$ that

$$2P^0(Y_t \geq y + 2\varepsilon) - 2 \sum_{k=1}^n P^0(Y_{\frac{tk}{n}} - Y_{\frac{t(k-1)}{n}} \geq \varepsilon) \leq P^0(\sup_{k \leq n} Y_{\frac{tk}{n}} \geq y).$$

Note that $\sum_{k=1}^n P^0(Y_{\frac{tk}{n}} - Y_{\frac{t(k-1)}{n}} \geq \varepsilon) = nP^0(Y_{\frac{t}{n}} \geq \varepsilon) \rightarrow t \int_{\varepsilon}^{\infty} \nu(x) dx$, hence, by symmetry again

$$\begin{aligned} P^0(\sup_{s \leq t} Y_s \geq y) &\geq 2P^0(Y_t \geq y + 2\varepsilon) - 2t \int_{\varepsilon}^{\infty} \nu(x) dx \\ &= P^0(|Y_t| \geq y + 2\varepsilon) - 2t \int_{\varepsilon}^{\infty} \nu(x) dx. \end{aligned}$$

This implies that

$$P^x(\tau_{\mathbb{H}} > t) = P^0(\sup_{s \leq t} Y_s < x_d) \leq P^0(|Y_t| < x_d + 2\varepsilon) + 2t \int_{\varepsilon}^{\infty} \nu(x) dx.$$

For $\varepsilon \geq 1$ we obtain from (2.9) and (2.6)

$$\int_{\varepsilon}^{\infty} \nu(x) dx \leq C e^{-\varepsilon} \varepsilon^{-\alpha/2-1}.$$

Lemma 2.1 implies that the density of $Y(t)$ is bounded by $Ct^{-1/2}$, $t \geq 1$, hence taking $\varepsilon = \frac{3}{2} \ln(t+1)$ we obtain

$$P^x(\tau_{\mathbb{H}} > t) \leq C(x_d + \ln(t+1)) t^{-1/2}.$$

□

In order to improve the above estimate for x close to the boundary we use Lemma 2.5.

Lemma 2.8. *For $0 < x_d < 2$ we have*

$$P^x(\tau_{\mathbb{H}} > t) \leq C x_d^{\alpha/2} \ln(t+1)/t^{1/2}, \quad t \geq 2, \quad (2.19)$$

where C is a constant.

Proof. It is enough to prove the claim for $d = 1$. Let $D = (0, 2)$ and assume that $0 < x < 2$.

By the Strong Markov Property and then by Lemma 2.7 we obtain for $t \geq 1$:

$$\begin{aligned} P^x(\tau_{(0,\infty)} > 2t) &\leq P^x(\tau_D > t, \tau_{(0,\infty)} > 2t) \\ &\quad + E^x[\tau_D < \tau_{(0,\infty)}; P^{X_{\tau_D}}(\tau_{(0,\infty)} > t)] \\ &\leq P^x(\tau_D > t) \\ &\quad + C E^x[\tau_D < \tau_{(0,\infty)}; X_{\tau_D} + \ln(t+1)]/t^{1/2} \\ &\leq \frac{E^x \tau_D}{t} + C E^x[X_{\tau_D} > 2; X_{\tau_D}]/t^{1/2} \\ &\quad + C \ln(t+1) P^x(X_{\tau_D} > 2)/t^{1/2} \\ &\leq C x^{\alpha/2} \ln(t+1)/t^{1/2}. \end{aligned}$$

The last inequality follows from Lemma 2.5. The proof is complete. □

The estimates from Lemmas 2.7, 2.8 will be very useful for the estimates of the Green function of the half-line, however they are not optimal. In the sequel we will be able to improve them to be sharp enough and optimal (see Proposition 4.5). This will have a great importance in estimating the Green function for a half-space in the d -dimensional case.

Lemma 2.9. *There is a constant C such that for any open set D :*

$$Eg_{T_\alpha(t)}^D(x, y) \leq p_t^D(x, y) \leq C(t^{-d/2} + t^{-d/\alpha})P^x(\tau_D > t/3)P^y(\tau_D > t/3),$$

where $g_t^D(x, y)$ is the transition probability for the Brownian motion killed on exiting D .

Proof. We start with the upper bound. Since $p_t^D(x, y)$ is a density of a semigroup and $p_t^D(x, y) \leq \max_{z \in \mathbb{R}^d} p_t(z)$ then we have

$$\begin{aligned} p_{3t}^D(x, y) &= \int_D \int_D p_t^D(x, z)p_t^D(z, w)p_t^D(w, y)dz dw \\ &\leq \max_{z \in \mathbb{R}^d} p_t(z) \int_D p_t^D(x, z)dz \int_D p_t^D(w, y)dw \\ &= \max_{z \in \mathbb{R}^d} p_t(z)P^x(\tau_D > t)P^y(\tau_D > t), \end{aligned}$$

which proves the upper bound since $\max_{z \in \mathbb{R}^d} p_t(z) \leq C(t^{-d/2} + t^{-d/\alpha})$ by Lemma 2.1.

To get the lower bound we use the subordination of the process to the Brownian motion: $X_t = B_{T_\alpha(t)}$. Then

$$\begin{aligned} p_t^D(x, y) &= P^x(B_{T_\alpha(t)} \in dy, B_{T_\alpha(s)} \in D, 0 \leq s < t) \\ &\geq P^x(B_{T_\alpha(t)} \in dy, B_s \in D, 0 \leq s < T_\alpha(t)). \end{aligned}$$

Using the independence of T_α and the Brownian motion B we obtain

$$P^x(B_{T_\alpha(t)} \in dy, B_s \in D, 0 \leq s < T_\alpha(t) | T_\alpha(\cdot)) = g_{T_\alpha(t)}^D(x, y),$$

Integrating we obtain the lower bound. □

The following lemma provides a very useful lower bound. Its proof closely follows the approach used in [20], where the bounds on the potential kernels (Green functions for the whole \mathbb{R}^d) were established for some special subordinated Brownian motions (in particular for our process) for $d \geq 3$.

Lemma 2.10. *For any open set $D \in \mathbb{R}^d$ we have*

$$G_D(x, y) \geq \frac{2}{\alpha} G_D^{gauss}(x, y),$$

where $G_D^{gauss}(x, y)$ is the Green function of D for the Brownian motion.

Proof. Let $Q(x, y) = \int_0^\infty Eg_{T_\alpha(t)}^D(x, y)dt$. From the previous lemma it is enough to prove that $Q(x, y) \geq \frac{2}{\alpha} G_D^{gauss}(x, y)$. We have

$$\begin{aligned} Q(x, y) &= \int_0^\infty Eg_{T_\alpha(t)}^D(x, y)dt \\ &= \int_0^\infty e^t \int_0^\infty g_u^D(x, y)e^{-u}\theta_\alpha(t, u)dudt \\ &= \int_0^\infty g_u^D(x, y)e^{-u} \int_0^\infty e^t\theta_\alpha(t, u)dtdu \\ &= \int_0^\infty g_u^D(x, y)G(u)du, \end{aligned}$$

where $G(u) = e^{-u} \int_0^\infty e^t \theta_\alpha(t, u) dt$ is the potential kernel of the subordinator $T_\alpha(t)$. It was proved in [20] that $G(u)$ is a completely monotone (hence decreasing) function and $\inf_{u>0} G(u) = \lim_{u \rightarrow \infty} G(u) = C_\alpha$. We find the constant C_α by taking into account the asymptotics of the Laplace transform of $G(u)$ at the origin:

$$\begin{aligned} \int_0^\infty e^{-\lambda u} G(u) du &= \int_0^\infty e^t \int_0^\infty e^{-u(1+\lambda)} \theta_\alpha(t, u) du dt \\ &= \int_0^\infty e^t e^{-(1+\lambda)^{\alpha/2} t} dt = \frac{1}{(1+\lambda)^{\alpha/2} - 1} \\ &\cong \frac{2}{\lambda \alpha}, \quad \lambda \rightarrow 0. \end{aligned}$$

Applying the monotone density theorem we obtain that $C_\alpha = 2/\alpha$. Thus, since $g_u^D(x, y) \geq 0$, we finally obtain

$$\begin{aligned} Q(x, y) &= \int_0^\infty g_u^D(x, y) G(u) du \\ &\geq \frac{2}{\alpha} \int_0^\infty g_u^D(x, y) du = \frac{2}{\alpha} G_D^{gauss}(x, y). \end{aligned}$$

□

At this point let us recall that the exact formulas for the Brownian Green functions are well known for several regular sets as balls or half-spaces (see e.g. [1]). Since some of them will be useful in the sequel we will list them for the future reference. Recall that the Brownian motion we refer to in this paper has its clock running twice faster than the usual Brownian motion. For the half-space \mathbb{H} , for $d \geq 3$, we have that

$$\begin{aligned} G_{\mathbb{H}}^{gauss}(x, y) &= C(d) \left[\frac{1}{|x - y|^{d-2}} - \frac{1}{|x - y^*|^{d-2}} \right] \\ &\approx \min \left\{ \frac{x_d y_d}{|x - y|^d}, \frac{1}{|x - y|^{d-2}} \right\}, \quad x, y \in \mathbb{H}, \end{aligned} \quad (2.20)$$

where $y^* = (y_1, \dots, y_{d-1}, -y_d) \in \mathbb{H}^c$.

For the half-space \mathbb{H} , for $d = 2$,

$$G_{\mathbb{H}}^{gauss}(x, y) = \frac{1}{2\pi} \ln \left(1 + 4 \frac{x_2 y_2}{|x - y|^2} \right), \quad x, y \in \mathbb{H}. \quad (2.21)$$

In the one dimensional case

$$G_{(0, \infty)}^{gauss}(x, y) = x \wedge y, \quad x, y > 0. \quad (2.22)$$

For the finite interval $(0, R)$ we have

$$G_{(0, R)}^{gauss}(x, y) = \frac{x(R - y) \wedge y(R - x)}{R}, \quad x, y \in (0, R). \quad (2.23)$$

Lemma 2.11. *Let D be an open subset of \mathbb{H} . For fixed $y \in \mathbb{H}$ the function $G_{\mathbb{H}}(\cdot, y)$ is regular harmonic on D provided $y \notin \overline{D}$. The same conclusion holds if \mathbb{H} is replaced by an open bounded set.*

Proof. The proof is standard and is included for completeness. First observe that $G_{\mathbb{H}}(z, w) < \infty$ for $z \neq w$, which follows from Lemmas 2.2, 2.7 and 2.9. Next, applying (2.14) with $D_2 = \mathbb{H}$ and $D_1 = D$ we have

$$G_{\mathbb{H}}^{\lambda}(x, y) - G_D^{\lambda}(x, y) = E^x [\tau_D < \tau_{\mathbb{H}}; e^{-\lambda\tau_D} G_{\mathbb{H}}^{\lambda}(X_{\tau_D}, y)]. \quad (2.24)$$

If $y \notin \overline{D}$ then $\tau_D = 0$ and $X_{\tau_D} = y$, P^y - a.s. so $G_D^{\lambda}(x, y) = G_D^{\lambda}(y, x) = 0$. Moreover $E^x G_{\mathbb{H}}^{\lambda}(X_{\tau_{\mathbb{H}}}, y) = 0$, which follows from the fact that $P^x(X_{\tau_{\mathbb{H}}} \in \mathbb{H}^c \setminus (\mathbb{H}^c)^r) = 0$, where $(\mathbb{H}^c)^r$ is a set of regular points of \mathbb{H}^c and for every $z \in (\mathbb{H}^c)^r$, $y \in \mathbb{R}^d$ we have $G_{\mathbb{H}}^{\lambda}(z, y) = 0$ (see [3]). This implies that (2.24) can be rewritten as

$$G_{\mathbb{H}}^{\lambda}(x, y) = E^x e^{-\lambda\tau_D} G_{\mathbb{H}}^{\lambda}(X_{\tau_D}, y).$$

Passing with $\lambda \rightarrow 0$ and observing that $G_{\mathbb{H}}^{\lambda} \nearrow G_{\mathbb{H}}$ we obtain the conclusion by the monotone convergence theorem.

The same arguments can be applied for any bounded set F , since there is a half-space containing F , which guarantees that the Green function $G_F(x, y) < \infty$ for $x \neq y$. \square

The estimates below following from Theorem 2.6 were proved in [6]. They turn out to be useful in the next sections.

Theorem 2.12. *Assume that $d = 1$ and $\alpha \geq 1$. When $|x - y| \geq 1 \wedge x \wedge y > 0$ we obtain*

$$G_{(0,\infty)}^1(x, y) \approx \frac{e^{-|x-y|}}{|x-y|^{1-\alpha/2}} (1 \wedge x \wedge y)^{\alpha/2},$$

while for $|x - y| < 1 \wedge x \wedge y$ we obtain

$$\begin{aligned} G_{(0,\infty)}^1(x, y) &\approx \log \left[2 \frac{1 \wedge x \wedge y}{|x-y|} \right], \quad \text{if } \alpha = 1, \\ G_{(0,\infty)}^1(x, y) &\approx (1 \wedge x \wedge y)^{\alpha-1}, \quad \text{if } \alpha > 1. \end{aligned}$$

In the remaining case, $\alpha < d$, we have

$$G_{\mathbb{H}}^1(x, y) \approx \frac{K_{(d-\alpha)/2}(|x-y|)}{|x-y|^{(d-\alpha)/2}} \left[\left(\frac{1 \wedge x_d \wedge y_d}{|x-y| \wedge 1} \right)^{\alpha/2} \wedge 1 \right].$$

Finally we state some basic scaling properties both for the Poisson kernel and the Green function. The proof employs the scaling property (2.3) and consists of elementary but tedious calculation hence is omitted.

Lemma 2.13 (Scaling Property). *Let D be an open subset of \mathbb{R}^d and $P_{D,m}$, $G_{D,m}$ be the Poisson kernel, or the Green function, respectively, for D for the process with parameter m . Then*

$$\begin{aligned} P_{D,m}(x, u) &= m^{d/\alpha} P_{m^{1/\alpha}D}(m^{1/\alpha}x, m^{1/\alpha}u), \quad x \in D, u \in D^c, \\ G_{D,m}(x, y) &= m^{(d-\alpha)/\alpha} G_{m^{1/\alpha}D}(m^{1/\alpha}x, m^{1/\alpha}y), \quad x \in D, y \in D. \end{aligned}$$

Thus, if D is a cone with vertex at 0 we obtain:

$$\begin{aligned} P_{D,m}(x, u) &= m^{d/\alpha} P_D(m^{1/\alpha}x, m^{1/\alpha}u), \quad x \in D, u \in D^c, \\ G_{D,m}(x, y) &= m^{(d-\alpha)/\alpha} G_D(m^{1/\alpha}x, m^{1/\alpha}y), \quad x \in D, y \in D. \end{aligned}$$

Due to these scaling properties it is enough to investigate the case $m = 1$.

3 Green function of half-line

In this section $d = 1$ and the half-space \mathbb{H} is a half-line, that is $\mathbb{H} = (0, \infty)$.

Lemma 3.1. *Assume that $|x - y| \leq 3$. Then there is $C = C(\alpha)$ such that*

$$(1 \wedge x \wedge y)^{\alpha/2} \leq CG_{(0,\infty)}^1(x, y).$$

Proof. We use Theorem 2.12. First let $\alpha \geq 1$ and $|x - y| \geq 1 \wedge x \wedge y > 0$, then

$$\begin{aligned} G_{(0,\infty)}^1(x, y) &\geq ce^{-|x-y|}|x - y|^{\alpha/2-1}(1 \wedge x \wedge y)^{\alpha/2} \\ &\geq ce^{-3}3^{\alpha/2-1}(1 \wedge x \wedge y)^{\alpha/2}. \end{aligned}$$

Suppose that $|x - y| < 1 \wedge x \wedge y$. For $\alpha = 1$,

$$G_{(0,\infty)}^1(x, y) \geq c \log \left[2 \frac{1 \wedge x \wedge y}{|x - y|} \right] \geq c \log 2 \geq c(1 \wedge x \wedge y)^{\alpha/2}.$$

For $\alpha > 1$,

$$G_{(0,\infty)}^1(x, y) \geq c(1 \wedge x \wedge y)^{\alpha-1} \geq c(1 \wedge x \wedge y)^{\alpha/2}.$$

Next, observe that $K_{(1-\alpha)/2}(r)/r^{(1-\alpha)/2}$ is decreasing. Therefore for $\alpha < 1$ we obtain

$$\begin{aligned} G_{(0,\infty)}^1(x, y) &\geq c \frac{K_{(1-\alpha)/2}(|x - y|)}{|x - y|^{(1-\alpha)/2}} \left[\left(\frac{1 \wedge x \wedge y}{|x - y| \wedge 1} \right)^{\alpha/2} \wedge 1 \right] \\ &\geq c \frac{K_{(1-\alpha)/2}(3)}{3^{(1-\alpha)/2}} (1 \wedge x \wedge y)^{\alpha/2}. \end{aligned}$$

□

Theorem 3.2. *For $x, y > 0$,*

$$G_{(0,\infty)}(x, y) \approx G_{(0,\infty)}^1(x, y) + (x \wedge y) \vee (x \wedge y)^{\alpha/2}.$$

Proof. Throughout the whole proof we assume that $0 < x \leq y$. The proof will rely on the estimates of $P^x(\tau_{(0,\infty)} > t)$ derived in the previous section and the application of Lemma 2.9.

We proceed to estimate the Green function from above. First we split the integration

$$\begin{aligned} \int_0^\infty p_t^{(0,\infty)}(x, y) dt &= \int_0^6 p_t^{(0,\infty)}(x, y) dt + \int_6^\infty p_t^{(0,\infty)}(x, y) dt \\ &= V(x, y) + R(x, y). \end{aligned}$$

We start with the estimation of the second integral. Due to Lemma 2.9,

$$\begin{aligned} R(x, y) &= 3 \int_2^\infty p_{3t}^{(0,\infty)}(x, y) dt \\ &\leq c \int_2^\infty P^x(\tau_{(0,\infty)} > t) P^y(\tau_{(0,\infty)} > t) \frac{dt}{t^{1/2}}. \end{aligned}$$

First consider the case $y < \sqrt{2}$. Then using (2.19) we have

$$R(x, y) \leq C(xy)^{\alpha/2} \int_2^\infty \frac{(\ln t)^2}{t} \frac{dt}{t^{1/2}} \leq C(xy)^{\alpha/2}.$$

If $y \geq \sqrt{2}$, using (2.18) we estimate

$$\begin{aligned}
R(x, y) &\leq c \int_2^\infty (P^y(\tau_{(0,\infty)} > t))^2 \frac{dt}{t^{1/2}} \\
&= c \int_2^{y^2} (P^y(\tau_{(0,\infty)} > t))^2 \frac{dt}{t^{1/2}} + c \int_{y^2}^\infty (P^y(\tau_{(0,\infty)} > t))^2 \frac{dt}{t^{1/2}} \\
&\leq C \int_2^{y^2} \frac{dt}{t^{1/2}} + C \int_{y^2}^\infty \left(\frac{y + \ln t}{t^{1/2}} \right)^2 \frac{dt}{t^{1/2}} \\
&\leq Cy.
\end{aligned}$$

Hence

$$G_{(0,\infty)}(x, y) \leq C \begin{cases} V(x, y) + (xy)^{\alpha/2}, & y < 1, \\ V(x, y) + y, & y \geq 1. \end{cases} \quad (3.1)$$

Let $B = (n+2, \infty)$, $n \in \mathbb{N}$. Now assume that $n < x \leq n+1$ and $y \in B$. We claim that

$$G_{(0,\infty)}(x, y) \leq Cx, \quad (3.2)$$

where C depends only on α .

Observe that,

$$\begin{aligned}
\int_0^\infty V(x, y) dy &= \int_0^6 \int_0^\infty p_{(0,\infty)}(t, x, y) dy dt \\
&= \int_0^6 P^x(\tau_{(0,\infty)} > t) dt \leq 6.
\end{aligned} \quad (3.3)$$

Consider $h(v) = G_{(0,\infty)}(x, v)$, $v \in B$. By Lemma 2.11 it is regular harmonic on B . Hence using the estimate (3.1) we obtain

$$\begin{aligned}
G_{(0,\infty)}(x, y) &= E^y G_{(0,\infty)}(x, X_{\tau_B}) \\
&= E^y [G_{(0,\infty)}(x, X_{\tau_B}); X_{\tau_B} \in (0, n+2)] \\
&\leq E^y V(x, X_{\tau_B}) + C(n+2).
\end{aligned}$$

Integrating $G_{(0,\infty)}(v, y)$ with respect to dv and applying (3.3) we obtain

$$\int_n^{n+1} G_{(0,\infty)}(v, y) dv \leq 6 + C(n+2). \quad (3.4)$$

The final argument for proving (3.2) will use Lemma 2.5. Take $D = (n-1, n+2)$, and recall that $y > n+2$ and $x \in (n, n+1)$. Due to Lemma 2.11 the Green function $G_{(0,\infty)}(u, y)$ is positive regular harmonic on D as a function of u . By Harnack's inequality for harmonic functions on D , which follows from Lemma 2.5, we arrive at

$$G_{(0,\infty)}(x, y) \leq C G_{(0,\infty)}(u, y), \quad x, u \in (n, n+1),$$

which together with (3.4) completes the proof of the estimate

$$G_{(0,\infty)}(x, y) \leq Cx, \quad 1 < x \leq y-2.$$

Combining this with (3.1) we obtain

$$G_{(0,\infty)}(x, y) \leq C(V(x, y) + x), \quad x \geq 1.$$

Since $G_{(0,\infty)}^{gauss}(x, y) = x$ (see (2.21)), then by Lemma 2.10 we have that

$$G_{(0,\infty)}(x, y) \geq \frac{2}{\alpha}x.$$

Therefore we proved that

$$G_{(0,\infty)}(x, y) \approx V(x, y) + x, \quad x \geq 1. \quad (3.5)$$

To estimate $V(x, y)$ we use

$$\begin{aligned} V(x, y) &= \int_0^6 p_t^{(0,\infty)}(x, y) dt \leq e^6 \int_0^6 e^{-t} p_t^{(0,\infty)}(x, y) dt \\ &\leq e^6 G_{(0,\infty)}^1(x, y). \end{aligned}$$

Next consider $x < 1$ and $y \leq 2$. By (3.1) and Lemma 3.1 we get

$$G_{(0,\infty)}(x, y) \approx G_{(0,\infty)}^1(x, y) \approx G_{(0,\infty)}^1(x, y) + x^{\alpha/2}.$$

Now assume that $x < 1$ and $y > 2$. Again $G_{(0,\infty)}(\cdot, y)$, by Lemma 2.11, is regular harmonic on $(0, 2)$, hence by BHP (see Lemma 2.4):

$$G_{(0,\infty)}(x, y) \approx G_{(0,\infty)}(1, y)x^{\alpha/2}.$$

Due to Theorem 2.12, $G_{(0,\infty)}^1(1, y) \leq C$ so by (3.5) we have

$$G_{(0,\infty)}(1, y) \approx 1,$$

which implies

$$G_{(0,\infty)}(x, y) \approx x^{\alpha/2}, \quad x < 1, y > 2.$$

This completes the proof. □

Remark 3.3. *Let $x \leq y$. Then we have*

$$G_{(0,\infty)}(x, y) \approx \begin{cases} G_{(0,\infty)}^1(x, y), & x \leq 1, |x - y| < 1; \\ G_{(0,\infty)}^1(x, y) + x, & x > 1, |x - y| < 1; \\ x \vee x^{\alpha/2}, & |x - y| \geq 1. \end{cases} \quad (3.6)$$

4 Exit time properties

In this section we derive optimal estimates of the expected value of the exit time from a ball of arbitrary radius. Then, which seems the most important result of this section, we provide optimal estimates of the tail distribution for the exit time from a half-space. That is we improve the bounds obtained in Lemmas 2.7 and 2.8. They will play a crucial role in the next section, where we deal with the Green function of a half-space in \mathbb{R}^d . We start with the one-dimensional case.

Proposition 4.1. *For $x \in (0, R)$ we have*

$$E^x \tau_{(0,R)} \approx (x^{\alpha/2} \vee x) \left((R - x)^{\alpha/2} \vee (R - x) \right).$$

Proof. If $R \leq 3$ then from Lemma 2.5 we have

$$E^x \tau_{(0,R)} \approx (x(R-x))^{\alpha/2}.$$

Throughout the rest of the proof we suppose that $R > 3$. Assume $x \leq R/2$. First we prove the upper bound. By Theorem 3.2 and (2.17) we obtain

$$\begin{aligned} E^x \tau_{(0,R)} &= \int_0^R G_{(0,R)}(x, y) dy \leq \int_0^R G_{(0,\infty)}(x, y) dy \\ &\approx \int_0^R G_{(0,\infty)}^1(x, y) dy + \int_0^x (y^{\alpha/2} \vee y) dy + (R-x)(x^{\alpha/2} \vee x) \\ &\leq 2(\alpha\Gamma(\alpha/2))^{-1} x^{\alpha/2} + x(x^{\alpha/2} \vee x) + (R-x)(x^{\alpha/2} \vee x) \\ &\leq cR(x^{\alpha/2} \vee x). \end{aligned} \tag{4.1}$$

Now, we deal with the lower bound. By Lemma 2.10,

$$G_{(0,R)}(x, y) \geq \frac{2}{\alpha} G_{(0,R)}^{gauss}(x, y).$$

Denote the first exit time of $(0, R)$ for the Brownian motion by $\tau_{(0,R)}^{gauss}$. It is well known that $E^x \tau_{(0,R)}^{gauss} = \frac{1}{2} x (R-x)$ (eg. see [10]). Then we have

$$\begin{aligned} E^x \tau_{(0,R)} &= \int_0^R G_{(0,R)}(x, y) dy \geq \frac{2}{\alpha} \int_0^R G_{(0,R)}^{gauss}(x, y) dy \\ &= \frac{2}{\alpha} E^x \tau_{(0,R)}^{gauss} = \frac{1}{\alpha} x (R-x). \end{aligned}$$

Hence we get, for $1 \leq x \leq R/2$,

$$E^x \tau_{(0,R)} \approx x R \tag{4.2}$$

Let $x < 1$. Notice that by the Strong Markov Property

$$E^x \tau_{(0,R)} = s(x) + E^x \tau_{(0,2)},$$

where $s(x) = E^x(E^{X_{\tau_{(0,2)}}} \tau_{(0,R)})$ is regular harmonic on the interval $(0, 2)$ vanishing on its complement. Therefore by BHP (see Lemma 2.4) we obtain

$$s(x) \approx s(1)x^{\alpha/2}.$$

Moreover due to Lemma 2.5 we have

$$E^x \tau_{(0,2)} \approx x^{\alpha/2}.$$

This yields

$$E^x \tau_{(0,R)} \approx (s(1) + 1)x^{\alpha/2}.$$

Noting that $s(1) = E^1 \tau_{(0,R)} - E^1 \tau_{(0,2)}$ and observing that (4.2) implies

$$(E^1 \tau_{(0,R)} - E^1 \tau_{(0,2)}) + 1 \approx R,$$

we obtain

$$E^x \tau_{(0,R)} \approx R x^{\alpha/2}, \quad 0 < x < 1. \tag{4.3}$$

Putting together (4.1), (4.2) and (4.3) we obtain

$$E^x \tau_{(0,R)} \approx R(x^{\alpha/2} \vee x), \quad \text{for } x \leq R/2.$$

By symmetry we have $E^x \tau_{(0,R)} = E^{R-x} \tau_{(0,R)}$, which ends the proof. \square

Now we derive bounds for the expected exit times from balls in the multidimensional case.

Proposition 4.2. *For $x \in B(0, R) = \{v \in \mathbb{R}^d : |v| < R\}$ we have*

$$E^x \tau_{B(0,R)} \approx ((R - |x|)^{\alpha/2} \vee (R - |x|)) (R \vee R^{\alpha/2}).$$

Proof. Let $\tau_{B(0,R)}^{stable}$ be the first exit time from $B(0, R)$ for the α -stable isotropic process. By the result of Gettoor [12] we have $E^x \tau_{B(0,R)}^{stable} = c(R^2 - |x|^2)^{\alpha/2}$ for $c = c(\alpha, d)$.

First assume that $R \leq 3$. Then from (2.16) we obtain

$$E^x \tau_{B(0,R)} \approx E^x \tau_{B(0,R)}^{stable} = c(R^2 - |x|^2)^{\alpha/2} \approx (R - |x|)^{\alpha/2} R^{\alpha/2},$$

which completes the proof in this case.

Next suppose that $R > 3$. Let $z = x/|x|$ if $x \neq 0$ and $z = (1, 0, \dots, 0)$ if $x = 0$. We now take $S_R = \{v : |\langle z, v \rangle| < R\}$. The process $\langle z, X_t \rangle$ is the one-dimensional relativistic process (with the same parameter) which starts from $|x|$. Note that

$$E^x \tau_{B(0,R)} \leq E^x \tau_{S_R}.$$

By the one-dimensional result (see Lemma 4.1) we get the upper bound.

For $|x| \leq R-1$ we get the lower bound by using Lemma 2.10 and the result for the Brownian motion: $E^x \tau_{B(0,R)}^{gauss} = \frac{1}{2d}(R^2 - |x|^2)$ (see [10]). Namely

$$\begin{aligned} E^x \tau_{B(0,R)} &= \int_{B(0,R)} G_{B(0,R)}(x, y) dy \geq \frac{2}{\alpha} \int_{B(0,R)} G_{B(0,R)}^{gauss}(x, y) dy \\ &= \frac{2}{\alpha} E^x \tau_{B(0,R)}^{gauss} = \frac{1}{\alpha d} (R^2 - |x|^2). \end{aligned} \quad (4.4)$$

To complete the proof we need to consider $R-1 \leq |x| \leq R$. The conclusion will follow in the usual way from BHP (see Lemma 2.4) and the bound above for $|x| = R-1$. We may and do assume that $x = (0, \dots, 0, |x|)$. Denote $x_0 = (0, \dots, 0, R-1)$ and $z_0 = (0, \dots, 0, R)$. Let

$$F = B(0, R) \cap B(z_0, 2)$$

and

$$s(x) = E^x E^{X(\tau_F)} \tau_{B(0,R)}.$$

Observe that $s(x)$ is a positive regular harmonic function on F satisfying the assumptions of the second part of Lemma 2.4 hence

$$s(x) \approx s(x_0)(R - |x|)^{\alpha/2}.$$

Next, by the Strong Markov Property

$$\begin{aligned} E^x \tau_{B(0,R)} &= s(x) + E^x \tau_F \geq s(x) + E^x \tau_{B(x_0,1)} \\ &\approx s(x_0)(R - |x|)^{\alpha/2} + E^x \tau_{B(x_0,1)}^{stable} \\ &\approx (s(x_0) + 1)(R - |x|)^{\alpha/2} \\ &= (E^{x_0} \tau_{B(0,R)} - E^{x_0} \tau_F + 1)(R - |x|)^{\alpha/2} \\ &\geq c R (R - |x|)^{\alpha/2}. \end{aligned} \quad (4.5)$$

The equivalence $E^x \tau_{B(x_0,1)} \approx E^x \tau_{B(x_0,1)}^{stable}$ follows from (2.16) and

$$E^{x_0} \tau_{B(0,R)} - E^{x_0} \tau_F + 1 \geq cR$$

follows from (4.4). Combining (4.4) and (4.5) we arrive at the desired lower bound. \square

Now, we recall the Ikeda-Watanabe formula [16] which provides a relationship between the Green function and the Poisson kernel. Assume $D \subset \mathbb{R}^d$ is a nonempty open set and E is a Borel set such that $\text{dist}(D, E) > 0$, then we have

$$P^x(X(\tau_D) \in E, \tau_D < \infty) = \int_D G_D(x, y) \nu(E - y) dy, \quad x \in D. \quad (4.6)$$

The following generalization of the Ikeda-Watanabe formula was proved in [18]:

$$P^x(X(\tau_D) \in E, t_1 < \tau_D < t_2) = \int_D \int_{t_1}^{t_2} p_t^D(x, y) dt \nu(E - y) dy, \quad (4.7)$$

where $0 \leq t_1 < t_2$, $x \in D$. For D which satisfies the outer cone property we have $P^x(X_{\tau_D} \in \partial D, \tau_D < \infty) = 0$ (see [18]). Therefore the above formulas are true for all sets $E \subset D^c$ for such D . In particular, for sets studied in this paper as balls or half-spaces, the process does not hit the boundary, when exiting a set.

As a consequence of formula (4.6) we have the following lemma which proof is omitted.

Lemma 4.3. *Let $D \subset \mathbb{R}^d$ be a bounded open set then*

$$P_D(x, z) \leq E^x \tau_D \sup_{v \in D} \nu(z - v), \quad z \in (\overline{D})^c, x \in D.$$

Moreover, if $\text{dist}(z, D) \geq 1$ then

$$P_D(x, z) \leq C E^x \tau_D e^{-\text{dist}(z, D)}.$$

Proposition 4.4. *For $0 < x < R$ we have*

$$P^x(\tau_{(0, R)} < \tau_{(0, \infty)}) \approx \frac{x^{\alpha/2} \vee x}{R^{\alpha/2} \vee R}.$$

Proof. Assume that $R \geq 1$ and $0 < x < R$. By Lemma 2.11 the function $G_{(0, \infty)}(\cdot, 2R)$ is regular harmonic on $(0, R)$, therefore by Remark 3.3 we obtain

$$\begin{aligned} Cx^{\alpha/2} \vee x &\geq G_{(0, \infty)}(x, 2R) = E^x G_{(0, \infty)}(X_{\tau_{(0, R)}}, 2R) \\ &\geq cE^x[X_{\tau_{(0, R)}} \wedge 2R; X_{\tau_{(0, R)}} > 0] \\ &\geq cRP^x(X_{\tau_{(0, R)}} > 0). \end{aligned} \quad (4.8)$$

Let $n \geq 3$, which we specify later. Again $G_{(0, \infty)}(\cdot, nR)$ is regular harmonic on $(0, R)$. Applying Remark 3.3 we have

$$\begin{aligned} G_{(0, \infty)}(x, nR) &= E^x G_{(0, \infty)}(X_{\tau_{(0, R)}}, nR) \\ &\leq C \left(nRP^x(X_{\tau_{(0, R)}} > 0) + E^x G_{(0, \infty)}^1(X_{\tau_{(0, R)}}, nR) \right). \end{aligned} \quad (4.9)$$

Moreover Lemma 4.3 and Theorem 2.12 imply

$$\begin{aligned}
& E^x G_{(0,\infty)}^1(X_{\tau_{(0,R)}}, nR) \\
&= \int_R^\infty G_{(0,\infty)}^1(v, nR) P_{(0,R)}(x, v) dv \\
&= \int_R^{(n-1)R} G_{(0,\infty)}^1(v, nR) P_{(0,R)}(x, v) dv \\
&\quad + \int_{(n-1)R}^\infty G_{(0,\infty)}^1(v, nR) P_{(0,R)}(x, v) dv \\
&\leq \sup_{R \leq v \leq (n-1)R} G_{(0,\infty)}^1(v, nR) P^x(X_{\tau_{(0,R)}} \in (R, (n-1)R)) \\
&\quad + \sup_{v \geq (n-1)R} P_{(0,R)}(x, v) \int_{(n-1)R}^\infty G_{(0,\infty)}^1(v, nR) dv \\
&\leq c P^x(X_{\tau_{(0,R)}} > 0) + C e^{-(n-2)R} E^x \tau_{(0,R)},
\end{aligned} \tag{4.10}$$

where $P_{(0,R)}(x, v)$ is the Poisson kernel for $(0, R)$ and by Lemma 4.3 it admits

$$P_{(0,R)}(x, v) \leq C E^x \tau_{(0,R)} e^{-(n-2)R}, \quad v \geq (n-1)R.$$

Using the (4.9) and (4.10) we arrive at

$$c n R P^x(X_{\tau_{(0,R)}} > 0) \geq G_{(0,\infty)}(x, nR) - C e^{-(n-2)R} E^x \tau_{(0,R)}.$$

By Lemma 4.1, $E^x \tau_{(0,R)} \approx R(x^{\alpha/2} \vee x)$, so Remark 3.3 implies

$$G_{(0,\infty)}(x, nR) - C e^{-(n-2)R} E^x \tau_{(0,R)} \geq (c - C R e^{-(n-2)R}) (x^{\alpha/2} \vee x).$$

Now we pick n independently of $R \geq 1$ and large enough so that $c - C R e^{-(n-2)R} \geq c/2$. This yields

$$P^x(\tau_{(0,R)} < \tau_{(0,\infty)}) \geq c \frac{x^{\alpha/2} \vee x}{R}. \tag{4.11}$$

Next, for $R < 1$ we use Lemma 2.5 to get

$$P^x(\tau_{(0,R)} < \tau_{(0,\infty)}) \approx \frac{x^{\alpha/2}}{R^{\alpha/2}}. \tag{4.12}$$

Combining (4.8), (4.11) and (4.12) ends the proof. \square

Now we can prove the main result of this section.

Theorem 4.5. *For $x > 0$ and $t \geq 1$,*

$$P^x(\tau_{(0,\infty)} > t) \approx \frac{x^{\alpha/2} \vee x}{t^{1/2}} \wedge 1.$$

Proof. Assume that $t \geq 1$. If $2x \geq t^{1/2}$ the upper bound is trivial. So we may assume that $2x < t^{1/2}$. We have

$$P^x(\tau_{(0,\infty)} > t) \leq P^x(\tau_{(0,R)} > t) + P^x(\tau_{(0,R)} < \tau_{(0,\infty)}).$$

Let $R > 2x$. By Chebyshev's inequality and Proposition 4.1 we obtain

$$P^x(\tau_{(0,R)} > t) \leq \frac{E^x \tau_{(0,R)}}{t} \approx \frac{(R \vee R^{\alpha/2})(x^{\alpha/2} \vee x)}{t}.$$

By the Lemma 4.4

$$P^x(\tau_{(0,R)} < \tau_{(0,\infty)}) \leq c \frac{x^{\alpha/2} \vee x}{R^{\alpha/2} \vee R}.$$

Setting $R = t^{1/2}$ we arrive at the upper bound.

Next, let us observe that by Lemma 2.9,

$$\begin{aligned} P^x(\tau_{(0,\infty)} > t) &= \int_0^\infty p_t^{(0,\infty)}(x, y) dy \geq \int_0^\infty E g_{T_\alpha(t)}^{(0,\infty)}(x, y) dy \\ &= E P^x(\tau_{(0,\infty)}^{gauss} > T_\alpha(t)). \end{aligned} \quad (4.13)$$

Let us observe that by Chebyshev's inequality and (2.1) for $\lambda = -1$ we have

$$P(T_\alpha(t) > 2t) = P(e^{T_\alpha(t)} > e^{2t}) \leq E e^{T_\alpha(t)} e^{-2t} = e^{t-2t} \leq e^{-1}.$$

That is $P(T_\alpha(t) \leq 2t) \geq 1 - e^{-1}$. Taking into account the fact that $P^x(\tau_{(0,\infty)}^{gauss} > t) \approx \frac{x}{t^{1/2}} \wedge 1$ we obtain from (4.13),

$$P^x(\tau_{(0,\infty)} > t) \geq c E \left(\frac{x}{T_\alpha(t)^{1/2}} \wedge 1 \right) \geq c \left(\frac{x}{t^{1/2}} \wedge 1 \right).$$

Now, let $x < 1$ then

$$\begin{aligned} P^x(\tau_{(0,\infty)} > t) &\geq E^x \left[X_{\tau_{(0,2)}} > 0; P^{X_{\tau_{(0,2)}}}(\tau_{(0,\infty)} > t) \right] \\ &\geq c E^x \left[X_{\tau_{(0,2)}} \geq 2; \frac{X_{\tau_{(0,2)}}}{t^{1/2}} \wedge 1 \right] \\ &\geq c \left(\frac{1}{t^{1/2}} \wedge 1 \right) P^x(X_{\tau_{(0,2)}} \geq 2). \end{aligned}$$

Hence by Lemma 4.4 we obtain

$$P^x(\tau_{(0,\infty)} > t) \geq c x^{\alpha/2} \frac{1}{t^{1/2}},$$

which completes the proof. \square

Corollary 4.6. *There exists a constant C such that, for $t > 0$ and $x, y \geq 1$,*

$$p_t^{(0,\infty)}(x, y) \leq C(t^{-1/2} + t^{-1/\alpha}) \left(\frac{x}{t^{1/2}} \wedge 1 \right) \left(\frac{y}{t^{1/2}} \wedge 1 \right). \quad (4.14)$$

For $t \geq 1$ and $x, y > 0$,

$$c t^{-1/2} \left(\frac{x}{t^{1/2}} \wedge 1 \right) \left(\frac{y}{t^{1/2}} \wedge 1 \right) e^{-\frac{|x-y|^2}{c_1 t}} \leq p_t^{(0,\infty)}(x, y),$$

where c, c_1 are some constants. Hence, for $x, y, t \geq 1$, satisfying $t \geq |x - y|^2$ we have the optimal bound

$$p_t^{(0,\infty)}(x, y) \approx t^{-1/2} \left(\frac{x}{t^{1/2}} \wedge 1 \right) \left(\frac{y}{t^{1/2}} \wedge 1 \right).$$

Proof. The upper bound immediately follows from Lemma 2.9 and Theorem 4.5.

Pick $0 < \beta < 1/2$ such that $(1 + 1/\beta)^{\alpha/2} - 2 = 1$ and let

$$A_t = \{\omega : \beta t < T_\alpha(t)(\omega) < 2t\}.$$

To obtain the lower bound we use again Lemma 2.9 to get

$$p_t^{(0,\infty)}(x, y) \geq E \left[g_{T_\alpha(t)}^{(0,\infty)}(x, y); A_t \right].$$

Next by a classical result

$$\begin{aligned} g_t^{(0,\infty)}(x, y) &= g_t(x - y) - g_t(x + y) = g_t(x - y) \left(1 - e^{-\frac{xy}{t}} \right) \\ &\geq g_t(x - y) \left(1 \wedge \frac{xy}{t} \right) \geq g_t(x - y) \left(\frac{x}{t^{1/2}} \wedge 1 \right) \left(\frac{y}{t^{1/2}} \wedge 1 \right). \end{aligned}$$

Hence

$$p_t^{(0,\infty)}(x, y) \geq ct^{-1/2} e^{-\frac{|x-y|^2}{4\beta t}} P(A_t) \left(\frac{x}{t^{1/2}} \wedge 1 \right) \left(\frac{y}{t^{1/2}} \wedge 1 \right).$$

Next we estimate $P(A_t^c)$. By Chebyshev's inequality and by (2.1) for $\lambda = 1/\beta$,

$$\begin{aligned} P(T_\alpha(t) < \beta t) &= P(e^{-(1/\beta)T_\alpha(t)} > e^{-t}) \leq e^t E e^{-(1/\beta)T_\alpha(t)} \\ &= e^{-((1+1/\beta)^{\alpha/2}-2)t} = e^{-t}. \end{aligned}$$

Similarly by (2.1) for $\lambda = -1$,

$$P(T_\alpha(t) > 2t) = P(e^{T_\alpha(t)} > e^{2t}) \leq e^{-2t} E e^{T_\alpha(t)} = e^{-t}.$$

Hence

$$P(A_t^c) \leq 2e^{-t},$$

which implies $\inf_{t>1} P(A_t) \geq 1 - 2e^{-1}$ and this ends the proof. \square

One of the drawbacks of the inequality in the above Corollary is that the right hand side does not depend on the distance $|x - y|$. The following result will be very useful in the next section and it does take into account the distance $|x - y|$.

Theorem 4.7. *Let $x, y \geq 1$ and $|x - y| \geq 1$. Then, for $t \leq |x - y|^2$,*

$$p_t^{(0,\infty)}(x, y) \leq C \left(\frac{xy}{|x - y|^2} \wedge 1 \right) (g_t(c(x - y)) + t \nu(c(x - y)))$$

where $c = 8\sqrt{2}$ and C is some constant. Moreover

$$\int_1^\infty t^{-d/2+1/2} p_t^{(0,\infty)}(x, y) dt \leq c(d, \alpha) \frac{xy}{|x - y|^d}.$$

Proof. Our arguments are based on the idea of proof of Theorem 4.2 in [18]. Throughout the whole proof we assume that $x, y \geq 1$ and $x \leq y - 1$.

We first consider the case $t \leq |x - y|^2/16$. The interval $(0, (x + y)/2)$ we denote by S and $(y - s, y + s)$ by $D(s)$. Let $0 < s < 1/8$, then $D(s) \in (0, \infty) \setminus S$. By the Strong Markov Property we obtain

$$\begin{aligned}
& \int_{D(s)} p_t^{(0, \infty)}(x, y) dz \\
&= P^x(X_t \in D(s), \tau_{(0, \infty)} > t) \\
&\leq P^x(\tau_S < t, X_{\tau_S} > 0, X_t \in D(s)) \\
&= E^x[\tau_S < t, X_{\tau_S} \in (0, \infty) \setminus S, P^{X(\tau_S)}(X_{t-\tau_S} \in D(s))] .
\end{aligned} \tag{4.15}$$

Let $A = (y - |x - y|/4, y + |x - y|/4)$ and $B = (0, \infty) \setminus (S \cup A)$. Observe that $\text{dist}(A, S) = |x - y|/4$ and $\text{dist}(B, D(s)) \geq |x - y|/8$. Because $p_t(x)$ is radially decreasing in $|x|$ we have for $X_{\tau_S} \in B$,

$$\begin{aligned}
P^{X(\tau_S)}(X_u \in D(s)) &= \int_{D(s)} p_u(X_{\tau_S} - z) dz \\
&\leq |D(s)| p_u\left(\frac{x - y}{8}\right) \\
&\leq c|D(s)| \left(g_u\left(\frac{x - y}{8\sqrt{2}}\right) + u\nu\left(\frac{x - y}{8\sqrt{2}}\right) \right),
\end{aligned}$$

where in the last step we applied Lemma 2.2. Next observe that $g_t(x)$ is an increasing function in t on the interval $(0, x^2/2)$. Hence, for $t \leq |x - y|^2/264$, we obtain, for $X_{\tau_S} \in B$,

$$P^{X(\tau_S)}(X_{t-\tau_S} \in D(s)) \leq c|D(s)| \left(g_t\left(\frac{x - y}{8\sqrt{2}}\right) + t\nu\left(\frac{x - y}{8\sqrt{2}}\right) \right).$$

Define $F(t, z) = g_t(z/(8\sqrt{2})) + t\nu(z/(8\sqrt{2}))$. Then Proposition 4.4 and the above estimate yield

$$\begin{aligned}
& E^x[\tau_S < t, X_{\tau_S} \in B, P^{X(\tau_S)}(X_{t-\tau_S} \in D(s))] \\
&\leq c|D(s)| F(t, x - y) P^x(\tau_S < t, X_{\tau_S} \in B) \\
&\leq c|D(s)| F(t, x - y) P^x(\tau_S < \tau_{(0, \infty)}) \\
&\leq c|D(s)| F(t, x - y) \frac{x}{x + y} \\
&\leq c|D(s)| F(t, x - y) \frac{xy}{|x - y|^2}.
\end{aligned} \tag{4.16}$$

For the set A we have by (4.7),

$$\begin{aligned}
& E^x[\tau_S < t, X_{\tau_S} \in A, P^{X(\tau_S)}(X_{t-\tau_S} \in D(s))] \\
&= \int_S \int_0^t p_r^S(x, z) \int_A \nu(z - w) P^w(X_{t-r} \in D(s)) dw dr dz.
\end{aligned}$$

Moreover

$$\begin{aligned}
\int_A P^w(X_t \in D(s)) dw &= \int_A \int_{D(s)} p(t, w, z) dz dw \\
&= \int_{D(s)} \left(\int_A p(t, w, z) dw \right) dz \\
&= \int_{D(s)} P^w(X_t \in A) dw \leq |D(s)|.
\end{aligned} \tag{4.17}$$

Using (4.17) and observing that $\nu(z - w) \leq \nu((x - y)/4)$, $w \in A, z \in S$ we obtain

$$\begin{aligned}
& E^x [\tau_S < t, X_{\tau_S} \in A, P^{X(\tau_S)}(X_{t-\tau_S} \in D(s))] \\
& \leq c|D(s)|\nu((x - y)/4) \int_S \int_0^t p_S(r, x, z) dr dz \\
& = c|D(s)|\nu((x - y)/4) \int_0^t P^x(\tau_S > r) dr \\
& \leq c|D(s)|t\nu((x - y)/4) \\
& \leq c|D(s)|t \frac{xy}{|x - y|^2} \nu((x - y)/(8\sqrt{2})), \tag{4.18}
\end{aligned}$$

where the last step follows from (2.9) and (2.6). Combining (4.15), (4.16) and (4.18) after dividing by $|D(s)|$ and passing $s \searrow 0$ we obtain for $x, y \geq 1$, $|x - y| \geq 1$ and $|x - y|^2 \geq 256t$,

$$p_t^{(0, \infty)}(x, y) \leq c \frac{xy}{|x - y|^2} \left(g_t \left((x - y)/(8\sqrt{2}) \right) + t\nu \left((x - y)/(8\sqrt{2}) \right) \right). \tag{4.19}$$

Next we consider $|x - y|^2 \leq 256t$. By (4.14) we get for $t \geq 1/256$ and $x, y \geq 1$,

$$p_t^{(0, \infty)}(x, y) \leq ct^{-1/2} \frac{xy}{t}. \tag{4.20}$$

Since for $t > |x - y|^2/256 \geq 1/256$,

$$ct^{-1/2} \leq g_t((x - y)/(8\sqrt{2}))$$

we obtain

$$p_t^{(0, \infty)}(x, y) \leq c \frac{xy}{t} g_t((x - y)/(8\sqrt{2})), \quad t > |x - y|^2/256.$$

The above inequality combined with (4.19), (4.20) and Lemma 2.2 implies the first claim of the theorem.

To prove the second conclusion of the theorem we apply (4.19) for $256t < |x - y|^2$ and (4.20) for $256t \geq |x - y|^2$ to get

$$\begin{aligned}
& \int_1^\infty t^{-(d-1)/2} p_t^{(0, \infty)}(x, y) dt \\
& \leq c \frac{xy}{|x - y|^2} \int_{1/256}^{|x-y|^2/256} t^{-(d-1)/2} \left(g_t \left(\frac{x - y}{8\sqrt{2}} \right) + t\nu \left(\frac{x - y}{8\sqrt{2}} \right) \right) dt \\
& \quad + cxy \int_{|x-y|^2/256}^\infty t^{-d/2-1} dt \\
& \leq cxy \left(\frac{|x - y|^{2-d}}{|x - y|^2} + \frac{e^{-|x-y|/16}}{|x - y|^{\alpha/2+3}} (1 \vee |x - y|^{5-d}) + \frac{1}{|x - y|^d} \right) \\
& \leq c \frac{xy}{|x - y|^d}.
\end{aligned}$$

Note that we used (2.6) to estimate the density of the Lévy measure. □

5 Green function of $\mathbb{H} \subset \mathbb{R}^d$, $d \geq 2$.

In this section we extend our one-dimensional estimates for a half-line to higher dimensions. To achieve this we start with some upper estimates of the transition densities of the killed process.

Note that by subordination we have

$$p_t(x) = E g_{T_\alpha(t)}(x), \quad x \in \mathbb{R}^d.$$

Let $A_t = \{\omega : \beta t < T_\alpha(t)(\omega) < 2t\}$ be the set defined in the proof of Corollary 4.6. Let us define

$$q_t(x) = E(g_{T_\alpha(t)}(x); A_t^c), \quad x \in \mathbb{R}^d.$$

In the sequel we will need a simple upper bound of $q(t, x)$. Note that $g_t(x) \leq \frac{c}{|x|^d}$, $t > 0$, and this used for $q_t(x)$ yields

$$q_t(x) \leq \frac{c}{|x|^d} P(A_t^c) \leq \frac{C}{|x|^d} e^{-2t}. \quad (5.1)$$

The next lemma will have a very important role in obtaining the upper bound for the Green function. We introduce the following notation. For $x \in \mathbb{R}^d$ we denote $\mathbf{x} = (x_1, \dots, x_{d-1})$ and by $\mathbf{g}_t(\mathbf{x})$ we denote the Brownian semigroup in \mathbb{R}^{d-1} .

Lemma 5.1. *There is a constant $C = C(d, \alpha)$ such that*

$$p_t^{\mathbb{H}}(x, y) \leq C \mathbf{g}_{2t}(\mathbf{x} - \mathbf{y}) p_t^{(0, \infty)}(x_d, y_d) + q_t(x - y), \quad x, y \in \mathbb{H}. \quad (5.2)$$

Proof. For $y \in \mathbb{H}$ and $\delta > 0$ denote $V = V_y(\delta) = [y, y + \delta] = \times_{i=1}^d [y_i, y_i + \delta] = \mathbf{V} \times V_d \subset \mathbb{H}$. Then by independence of the subordinator $T_\alpha(t)$ and Brownian motion B_t one gets

$$\begin{aligned} & P^x(X_t \in V, \tau_{\mathbb{H}} > t) \\ &= P^x(X_t \in V, \tau_{\mathbb{H}} > t, A_t) + P^x(X_t \in V, \tau_{\mathbb{H}} > t, A_t^c) \\ &= E \left[A_t; P^{\mathbf{x}}(\mathbf{B}_{T_\alpha(t)} \in \mathbf{V} | T_\alpha(\cdot)) \times \right. \\ &\quad \times P^{x_d} \left(B^{(d)}_{T_\alpha(t)} \in V_d, B^{(d)}_{T_\alpha(s)} > 0; 0 < s < t | T_\alpha(\cdot) \right) \Big] \\ &\quad + P^x(X_t \in V, \tau_{\mathbb{H}} > t, A_t^c) \\ &\leq \sup_{\beta t \leq u \leq 2t} P^{\mathbf{x}}(\mathbf{B}_u \in \mathbf{V}) P^{x_d}(B^{(d)}_{T_\alpha(t)} \in V_d, B^{(d)}_{T_\alpha(s)} > 0 : 0 < s < t) \\ &\quad + \int_V q_t(x - z) dz \\ &\leq C P^{\mathbf{x}}(\mathbf{B}_{2t} \in \mathbf{V}) P^{x_d}(X_t^{(d)} \in V_d, \tau_{\mathbb{H}} > t) + \int_V q(t, x - z) dz. \end{aligned}$$

After dividing both sides by $|V|$ and passing $\delta \searrow 0$ we obtain the conclusion. \square

Note that for any $x, y \in \mathbb{H}$ we can estimate $\mathbf{g}_{2t}(\mathbf{x} - \mathbf{y}) \leq ct^{-(d-1)/2}$ so from (5.2) we deduce that

$$p_t^{\mathbb{H}}(x, y) \leq ct^{-(d-1)/2} p_t^{(0, \infty)}(x_d, y_d) + q_t(x - y), \quad (5.3)$$

which will be well estimated with the help of Theorem 4.7.

Lemma 5.1, the estimate (5.1) and Theorem 4.7 show that for the points $x, y \in \mathbb{H}$ away from the boundary such that $|x - y| > 2$ the Green functions for the relativistic process and the Brownian motion are comparable. In view of the one-dimensional case this result, proved below, is not surprising.

Theorem 5.2. For $|x - y| > 2$ and $x_d, y_d \geq 1$ we have

$$G_{\mathbb{H}}(x, y) \approx G_{\mathbb{H}}^{gauss}(x, y).$$

Proof. The lower bound follows from Lemma 2.10.

We claim that the following upper bound holds:

$$G_{\mathbb{H}}(x, y) \leq c \frac{x_d y_d}{|x - y|^d}. \quad (5.4)$$

By (4.14),

$$p_t^{(0, \infty)}(x_d, y_d) \leq C \frac{x_d y_d}{t^{3/2}}, \quad t \geq 1,$$

which together with (5.2) and (5.1) yield the following bound for the transition density

$$p_t^{\mathbb{H}}(x, y) \leq C \mathbf{g}_{2t}(\mathbf{y} - \mathbf{x}) \frac{x_d y_d}{t^{3/2}} + c e^{-2t} |x - y|^{-d}, \quad t \geq 1.$$

Integrating it over $(1, \infty)$ we arrive at

$$\int_1^\infty p_t^{\mathbb{H}}(x, y) dt \leq C \frac{x_d y_d}{|\mathbf{y} - \mathbf{x}|^d} + \frac{c}{|x - y|^d} \leq C_1 \frac{x_d y_d}{|\mathbf{y} - \mathbf{x}|^d}. \quad (5.5)$$

If $|x_d - y_d| \geq 1$ we apply (5.3), (5.1) and Theorem 4.7 to arrive at

$$\begin{aligned} \int_1^\infty p_t^{\mathbb{H}}(x, y) dt &\leq C \int_1^\infty t^{-(d-1)/2} p_t^{(0, \infty)}(x_d, y_d) dt + \frac{c}{|x - y|^d} \\ &\leq C \frac{x_d y_d}{|x_d - y_d|^d}. \end{aligned} \quad (5.6)$$

Next note that by Lemma 2.2 we can estimate

$$\int_0^1 p_t^{\mathbb{H}}(x, y) dt \leq \int_0^1 p_t(x - y) dt \leq \frac{c}{|x - y|^d}.$$

This combined with (5.5) and (5.6) implies (5.4).

Now let $d \geq 3$. Since (see (2.12)),

$$G_{\mathbb{H}}(x, y) \leq C \frac{1}{|x - y|^{d-2}}$$

we have the following bound for $|x - y| > 2$,

$$G_{\mathbb{H}}(x, y) \leq C \min \left\{ \frac{x_d y_d}{|x - y|^d}, \frac{1}{|x - y|^{d-2}} \right\} \approx G_{\mathbb{H}}^{gauss}(x, y),$$

where the last equivalence follows from (2.20). This completes the proof in this case.

Now we finish the proof for $d = 2$. By (5.4), for $\frac{x_2 y_2}{|x - y|^2} \leq 1$, we have

$$G_{\mathbb{H}}(x, y) \leq C \frac{x_2 y_2}{|x - y|^2} \approx \ln \left(1 + 4 \frac{x_2 y_2}{|x - y|^2} \right) = 2\pi G_{\mathbb{H}}^{gauss}(x, y),$$

where the last equality is just (2.21). If $\frac{x_2 y_2}{|x-y|^2} > 1$, using Lemmas 2.2 and 2.9 together with Theorem 4.5 we obtain

$$\begin{aligned}
G_{\mathbb{H}}(x, y) &\leq \int_0^{|x-y|^2} p_t(x-y) dt \\
&\quad + c \int_{|x-y|^2}^{\infty} t^{-1} P^x(\tau_{\mathbb{H}} > t/3) P^y(\tau_{\mathbb{H}} > t/3) dt \\
&\leq \int_0^{|x-y|^2} \frac{c}{|x-y|^2} dt + C \int_{|x-y|^2}^{x_2 y_2} t^{-1} dt + C x_2 y_2 \int_{x_2 y_2}^{\infty} t^{-2} dt \\
&\leq c + C \ln \left(\frac{x_2 y_2}{|x-y|^2} \right) \\
&\leq C \ln \left(1 + 4 \frac{x_2 y_2}{|x-y|^2} \right) = 2C \pi G_{\mathbb{H}}^{gauss}(x, y),
\end{aligned}$$

which completes the proof for $d = 2$. □

Now we are ready to prove the main result of this section.

Theorem 5.3. *For $d \geq 3$ and $x, y \in \mathbb{H}$:*

$$G_{\mathbb{H}}(x, y) \approx \min \left\{ \frac{(x_d \vee x_d^{\alpha/2})(y_d \vee y_d^{\alpha/2})}{|x-y|^d}, \frac{1}{|x-y|^{d-2}} \right\}, \quad |x-y| > 3,$$

$$G_{\mathbb{H}}(x, y) \approx \left[\left(\frac{x_d \wedge y_d}{|x-y|} \right)^{\alpha/2} \wedge 1 \right] \frac{1}{|x-y|^{d-\alpha}}, \quad |x-y| \leq 3.$$

For $d = 2$ and $x, y \in \mathbb{H}$:

$$G_{\mathbb{H}}(x, y) \approx \ln \left(1 + 4 \frac{(x_2 \vee x_2^{\alpha/2})(y_2 \vee y_2^{\alpha/2})}{|x-y|^2} \right), \quad |x-y| > 3,$$

$$G_{\mathbb{H}}(x, y) \approx \left[\left(\frac{x_2 \wedge y_2}{|x-y|} \right)^{\alpha/2} \wedge 1 \right] \frac{1}{|x-y|^{2-\alpha}} + \ln(1 \vee (x_2 \wedge y_2)), \quad |x-y| \leq 3.$$

Proof. First assume $|x-y| \leq 3$. In the paper [5] it was proved

$$G_{\mathbb{H}}(x, y) \approx G_{\mathbb{H}}^1(x, y), \quad d \geq 3$$

and

$$G_{\mathbb{H}}(x, y) \approx G_{\mathbb{H}}^1(x, y) + \ln(1 \vee (x_2 \wedge y_2)), \quad d = 2.$$

Since $|x-y| \leq 3$ by Theorem 2.12 we obtain

$$G_{\mathbb{H}}(x, y) \approx \left[\left(\frac{x_d \wedge y_d}{|x-y|} \right)^{\alpha/2} \wedge 1 \right] \frac{1}{|x-y|^{d-\alpha}}, \quad d \geq 3$$

and

$$G_{\mathbb{H}}(x, y) \approx \left[\left(\frac{x_2 \wedge y_2}{|x-y|} \right)^{\alpha/2} \wedge 1 \right] \frac{1}{|x-y|^{2-\alpha}} + \ln(1 \vee (x_2 \wedge y_2)), \quad d = 2.$$

This yields the bound in the case $|x - y| \leq 3$.

We introduce the following notation $\tilde{x} = (x_1, \dots, x_{d-1}, 1 \vee x_d)$, $x^* = (x_1, \dots, x_{d-1}, 0)$. Now assume $|x - y| > 3$ and observe that implies that $|x - y| \approx |\tilde{x} - \tilde{y}| > 2$.

Then if both points are away from the boundary ($x_d \wedge y_d \geq 1$) we use Lemma 5.2 to have

$$G_{\mathbb{H}}(x, y) \approx G_{\mathbb{H}}^{gauss}(x, y) = G_{\mathbb{H}}^{gauss}(\tilde{x}, \tilde{y}).$$

Next suppose that $x_d < 1 \leq y_d$. Let $D(x^*) = B(x^*, \sqrt{2}) \cap \mathbb{H}$. Then $y \notin D(x^*)$ and $G_{\mathbb{H}}(\cdot, y)$ is a regular harmonic function on $D(x^*)$ vanishing on \mathbb{H}^c . Hence by BHP (see Lemma 2.4) and next by Theorem 5.2 we have

$$G_{\mathbb{H}}(x, y) \approx x_d^{\alpha/2} G_{\mathbb{H}}(\tilde{x}, y) = x_d^{\alpha/2} G_{\mathbb{H}}(\tilde{x}, \tilde{y}) \approx x_d^{\alpha/2} G_{\mathbb{H}}^{gauss}(\tilde{x}, \tilde{y}).$$

A similar argument applies for $x_d, y_d < 1$. Notice that $x \notin D(y^*)$ and $y \notin D(x^*)$. Hence $G_{\mathbb{H}}(\cdot, y)$ and $G_{\mathbb{H}}(x, \cdot)$ are regular harmonic function on $D(x^*)$ and $D(y^*)$, respectively, vanishing on \mathbb{H}^c . Hence Lemma 2.4 and Theorem 5.2 imply that

$$G_{\mathbb{H}}(x, y) \approx x_d^{\alpha/2} y_d^{\alpha/2} G_{\mathbb{H}}(\tilde{x}, \tilde{y}) \approx x_d^{\alpha/2} y_d^{\alpha/2} G_{\mathbb{H}}^{gauss}(\tilde{x}, \tilde{y}).$$

Taking into account all cases we have

$$G_{\mathbb{H}}(x, y) \approx (1 \wedge x_d)^{\alpha/2} (1 \wedge y_d)^{\alpha/2} G_{\mathbb{H}}^{gauss}(\tilde{x}, \tilde{y}), \quad |x - y| > 3.$$

Applying (2.20) and (2.21) we can rewrite the above bound as

$$G_{\mathbb{H}}(x, y) \approx (1 \wedge x_d)^{\alpha/2} (1 \wedge y_d)^{\alpha/2} \min \left\{ \frac{(x_d \vee 1)(y_d \vee 1)}{|\tilde{x} - \tilde{y}|^d}, \frac{1}{|\tilde{x} - \tilde{y}|^{d-2}} \right\},$$

for $d \geq 3$, and

$$G_{\mathbb{H}}(x, y) \approx (1 \wedge x_d)^{\alpha/2} (1 \wedge y_d)^{\alpha/2} \ln \left(1 + 4 \frac{(x_d \vee 1)(y_d \vee 1)}{|\tilde{x} - \tilde{y}|^2} \right),$$

for $d = 2$. Taking into account $|x - y| \approx |\tilde{x} - \tilde{y}| \geq 1$, for $|x - y| > 3$, we finally arrive at

$$G_{\mathbb{H}}(x, y) \approx \min \left\{ \frac{(x_d \vee x_d^{\alpha/2})(y_d \vee y_d^{\alpha/2})}{|x - y|^d}, \frac{1}{|x - y|^{d-2}} \right\}, \quad d \geq 3$$

and

$$G_{\mathbb{H}}(x, y) \approx \ln \left(1 + 4 \frac{(x_2^{\alpha/2} \vee x_2)(y_2^{\alpha/2} \vee y_2)}{|x - y|^2} \right), \quad d = 2.$$

□

Now we compare the Green functions for half-space for the relativistic process and for the corresponding stable process, so we recall the formula of the Green function in the stable case (see [4]):

$$G_{\mathbb{H}}^{stable}(x, y) = C(\alpha, d) |x - y|^{\alpha-d} \int_0^{\frac{4x_d y_d}{|x-y|^2}} \frac{t^{\alpha/2-1}}{(t+1)^{d/2}} dt.$$

One can derive sharp estimates from the above formula. Our results from Theorem 5.3, Theorem 3.2 and Theorem 2.12 show that for the points $x, y \in \mathbb{H}$ such that $|x - y| \leq 2$ the Green functions of the half-space \mathbb{H} for the relativistic process and for the corresponding stable process are comparable if $d \geq 3$. If $d = 1$ or $d = 2$ they are also comparable but we have to assume additionally that the points are near the boundary.

Remark 5.4. Suppose that $|x - y| \leq 2$ then

$$G_{\mathbb{H}}(x, y) \approx \begin{cases} G_{\mathbb{H}}^{stable}(x, y), & d \geq 3; \\ G_{\mathbb{H}}^{stable}(x, y) + \ln(1 \vee (x_2 \wedge y_2)), & d = 2; \\ G_{\mathbb{H}}^{stable}(x, y) + (x \wedge y) \vee (x \wedge y)^{\alpha/2}, & d = 1. \end{cases}$$

From the estimates obtained in Theorem 5.3 and Theorem 3.2 we can infer that

$$G_{\mathbb{H}}(x, y) \geq C(G_{\mathbb{H}}^{stable}(x, y) + G_{\mathbb{H}}^{gauss}(x, y)), \quad x, y \in \mathbb{H}.$$

6 Green functions for intervals

In this section we provide optimal estimates for Green functions of bounded intervals. We know that for any interval the Green function is comparable with the corresponding Green function of the symmetric process. That is for the interval $(0, R)$, for $R \leq R_0$, we have

$$C(R_0)^{-1} G_{(0,R)}^{stable}(x, y) \leq G_{(0,R)}(x, y) \leq C(R_0) G_{(0,R)}^{stable}(x, y), \quad (6.1)$$

where $0 < x, y < R$. However, if R_0 grows, then the constant $C(R_0)$ tends to ∞ , so the above bound is not optimal in general case. The aim of this section is to provide optimal bounds for large intervals. We recall known estimates for stable cases:

$$G_{(0,R)}^{stable}(x, y) \approx \begin{cases} \min \left\{ \frac{1}{|x-y|^{1-\alpha}}, \frac{(\delta_R(x)\delta_R(y))^{\alpha/2}}{|x-y|} \right\}, & \alpha < 1, \\ \ln \left(1 + \frac{(\delta_R(x)\delta_R(y))^{1/2}}{|x-y|} \right), & \alpha = 1, \\ \min \left\{ (\delta_R(x)\delta_R(y))^{(\alpha-1)/2}, \frac{(\delta_R(x)\delta_R(y))^{\alpha/2}}{|x-y|} \right\}, & \alpha > 1, \end{cases} \quad (6.2)$$

where $\delta_R(x) = x \wedge (R - x)$.

We start with the proposition showing that for points x, y in the first half of the interval the Green function of the interval and the Green function of $(0, \infty)$ are comparable.

Proposition 6.1. Let $R \geq 4$. For $x, y \leq R/2 + 1$ we have

$$G_{(0,R)}(x, y) \approx G_{(0,\infty)}(x, y).$$

Proof. Throughout the whole proof we assume that $x \leq y \leq R/2 + 1$. Notice that it is enough to prove that

$$G_{(0,R)}(x, y) \geq c G_{(0,\infty)}(x, y),$$

for $x < 1$ or $|x - y| < 1$. Indeed, by Lemma 2.10 and Remark 3.3 we obtain for $x \geq 1$ and $|x - y| \geq 1$,

$$G_{(0,R)}(x, y) \geq \frac{2}{\alpha} G_{(0,R)}^{gauss}(x, y) = \frac{2}{\alpha} x(1 - y/R) \geq \frac{1}{2\alpha} x \geq c G_{(0,\infty)}(x, y).$$

We claim that

$$G_{(0,R)}(x, y) \geq C G_{(0,\infty)}^1(x, y), \quad |x - y| < 2. \quad (6.3)$$

Applying (2.14) with $D_2 = (0, \infty)$ and $D_1 = (0, R)$ we have

$$\begin{aligned} & G_{(0,\infty)}^1(x, y) - G_{(0,R)}^1(x, y) \\ &= E^x \left[\tau_{(0,R)} < \tau_{(0,\infty)}; e^{-\tau_{(0,R)}} G_{(0,\infty)}^1(X_{\tau_{(0,R)}}, y) \right] \\ &\leq \sup_{z \geq R} G_{\mathbb{H}}^1(z, y) P^x(\tau_{(0,R)} < \tau_{(0,\infty)}). \end{aligned}$$

Next, by Theorem 2.12 and Proposition 4.4 we obtain

$$G_{(0,\infty)}^1(x, y) - G_{(0,R)}^1(x, y) \leq Ce^{-R/2} R^{-2+\alpha/2} (x^{\alpha/2} \vee x).$$

Hence Lemma 3.1 yields

$$G_{(0,\infty)}^1(x, y) - G_{(0,R)}^1(x, y) \leq Ce^{-R/2} R^{-1+\alpha/2} G_{(0,\infty)}^1(x, y).$$

This proves (6.3) for $R > R_0$, if $R_0 > 4$ is large enough.

To handle the case $4 \leq R \leq R_0$ we apply (6.1) together with (6.2) and Theorem 2.12 to obtain

$$G_{(0,R)}(x, y) \geq c(R_0) G_{(0,R)}^{stable}(x, y) \geq cG_{(0,\infty)}^1(x, y),$$

which ends the proof of (6.3).

That is, by Theorem 3.2, for $|x - y| < 2$, $x \geq 1$, we get

$$\begin{aligned} G_{(0,R)}(x, y) &\geq c(G_{(0,R)}^1(x, y) + G_{(0,R)}^{gauss}(x, y)) \\ &\geq c(G_{(0,\infty)}^1(x, y) + x) \geq cG_{(0,\infty)}(x, y). \end{aligned} \quad (6.4)$$

Next, for $x < 1$ and $y \geq 2$, by BHP (Lemma 2.4),

$$\begin{aligned} G_{(0,R)}(x, y) &\geq cG_{(0,R)}(1, y)x^{\alpha/2} \geq cx^{\alpha/2}G_{(0,\infty)}(1, y) \\ &\geq cG_{(0,\infty)}(x, y). \end{aligned} \quad (6.5)$$

Combining (6.4) and (6.5) give us

$$G_{(0,R)}(x, y) \approx G_{(0,\infty)}(x, y),$$

for $x, y \leq R/2 + 1$. □

Proposition 6.2. *Let $R \geq 4$. Suppose that $1 \leq x \leq y \leq R - 1$, and $|x - y| \geq 1$ then*

$$G_{(0,R)}(x, y) \approx G_{(0,R)}^{gauss}(x, y) = x(R - y)/R.$$

Proof. Due to Lemma 2.10 we only need to prove upper bound. Suppose that $1 \leq x \leq y \leq R - 1$ and $|x - y| \geq 1$.

At first, let additionally $y \leq 3/4R$, then by Remark 3.3,

$$G_{(0,R)}(x, y) \leq G_{(0,\infty)}(x, y) \leq cx \leq 4cx(R - y)/R. \quad (6.6)$$

By symmetry and the above inequality we have

$$\begin{aligned} G_{(0,R)}(x, y) &= G_{(-R,0)}(-x, -y) = G_{(0,R)}(R - x, R - y) \\ &\leq 4c(R - y)x/R, \end{aligned} \quad (6.7)$$

for $R/4 \leq x \leq R/2 \leq y \leq R - 1$ and $|x - y| > 1$.

Hence it remains to consider the case $1 \leq x \leq R/4$ and $3R/4 \leq y \leq R - 1$. Denote $\eta = \tau_{(0,R/2)}$. Since $G_{(0,R)}(\cdot, y)$ is regular harmonic on $(0, R/2)$, so by Proposition 6.1 and

Remark 3.3,

$$\begin{aligned}
G_{(0,R)}(x, y) &= E^x G_{(0,R)}(X_\eta, y) \leq E^x [X_\eta > R/2; G_{(0,\infty)}(R - X_\eta, R - y)] \\
&\leq cE^x [X_\eta > R/2, |X_\eta - y| < 1; G_{(0,\infty)}^1(R - X_\eta, R - y)] \\
&\quad + cE^x [X_\eta > R/2; ((R - X_\eta) \vee (R - X_\eta)^{\alpha/2}) \wedge (R - y)] \\
&\leq c(R - y)P^x(\eta < \tau_{(0,\infty)}) \\
&\quad + cE^x [|X_\eta - y| < 1; G_{(0,\infty)}^1(R - X_\eta, R - y)] \\
&\leq c(R - y)x/R \\
&\quad + cE^x [|X_\eta - y| < 1; G_{(0,\infty)}^1(R - X_\eta, R - y)], \tag{6.8}
\end{aligned}$$

where the last inequality is a consequence of Proposition 4.4. Moreover, by Lemma 4.3 we obtain

$$\begin{aligned}
&E^x [|X_\eta - y| < 1; G_{(0,\infty)}^1(R - X_\eta, R - y)] \\
&= \int_{y-1}^{y+1} P_{(0,R/2)}(x, z) G_{(0,\infty)}^1(R - z, R - y) dz \\
&\leq cE^x \eta \int_{y-1}^{y+1} e^{-(z-R/2)} G_{(0,\infty)}^1(R - z, R - y) dz \\
&\leq c \frac{R}{2} x e^{-(y-1-R/2)} \int_0^\infty G_{(0,\infty)}^1(R - z, R - y) dz \\
&\leq cx(R - y)/R, \tag{6.9}
\end{aligned}$$

because $y \geq 3/4R$.

Combining (6.6), (6.7), (6.8) and (6.9) we obtain

$$G_{(0,R)}(x, y) \leq cx(R - y)/R = cG_{(0,R)}^{gauss}(x, y).$$

□

Now we can prove the main result of this section.

Theorem 6.3. *Let $R \geq 4$ and $x \leq y$ then we have for $|x - y| \leq 1$,*

$$G_{(0,R)}(x, y) \approx \min\{G_{(0,\infty)}(x, y), G_{(0,\infty)}(R - x, R - y)\}$$

and for $|x - y| > 1$

$$G_{(0,R)}(x, y) \approx \frac{(x^{\alpha/2} \vee x)((R - y)^{\alpha/2} \vee (R - y))}{R}.$$

Proof. Observe that by symmetry

$$G_{(0,R)}(x, y) = G_{(0,R)}(R - x, R - y). \tag{6.10}$$

The case $|x - y| \leq 1$ follows immediately from Proposition 6.1, Theorem 3.2 and (6.10).

For $0 < x \leq y < R$, $|x - y| > 1$ we define $\tilde{x} = x \vee 1$ and $\tilde{y} = y \wedge (R - 1)$. Then we can repeat the arguments used in the proof of Theorem 5.3 to arrive at

$$\begin{aligned} G_{(0,R)}(x, y) &\approx (1 \wedge x)^{\alpha/2} (1 \wedge (R - y))^{\alpha/2} G_{(0,R)}^{gauss}(\tilde{x}, \tilde{y}) \\ &= (1 \wedge x)^{\alpha/2} (1 \wedge (R - y))^{\alpha/2} \frac{\tilde{x}(R - \tilde{y})}{R} \\ &= \frac{(x^{\alpha/2} \vee x)((R - y)^{\alpha/2} \vee (R - y))}{R}, \quad |x - y| > 1. \end{aligned}$$

This completes the proof. □

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